Ordering Default Theories

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Comparing different theories: first-order logic

- In first-order logic, a theory T_1 is stronger than another theory T_2 if every formula derived from T_2 is also derived from T_1, i.e.

  $$T_1 \models T_2 \text{ and } T_2 \not\models T_1$$

- For instance, $T_1 = \{ \textit{bird}, \ \textit{bird} \rightarrow \textit{flies} \}$ is stronger than $T_2 = \{ \textit{bird} \vee \textit{flies} \}$.

Comparing different theories: Default Logic

$$T_1 = \{\ bird,\quad bird \rightarrow flies\ \},$$

$$T_3 = \{\ bird,\quad \frac{bird : flies}{flies}\ \}$$

- The fact *flies* derived from T_1 is a definite conclusion, while *flies* from T_3 is a default conclusion.
- Two theories have the same extension, but conclusions derived from T_1 are stronger than those of T_3.

Ordering Default Theories: Motivation

- In FOL theories are ordered by logical entailment. In DL extensions of theories are not helpful for judging relative strength between theories.
- To order default theories based on their information contents, it is necessary to distinguish different sorts of information derived from a theory.

Ordering Default Theories: Effect

- Comparing information contents of different theories is important to know relative value between theories.
- When there exist multiple sources of information as in multi-agent systems, it provides a yardstick for preference.
- It has potential application to the theory of induction in nonmonotonic logics and nonmonotonic ILP.

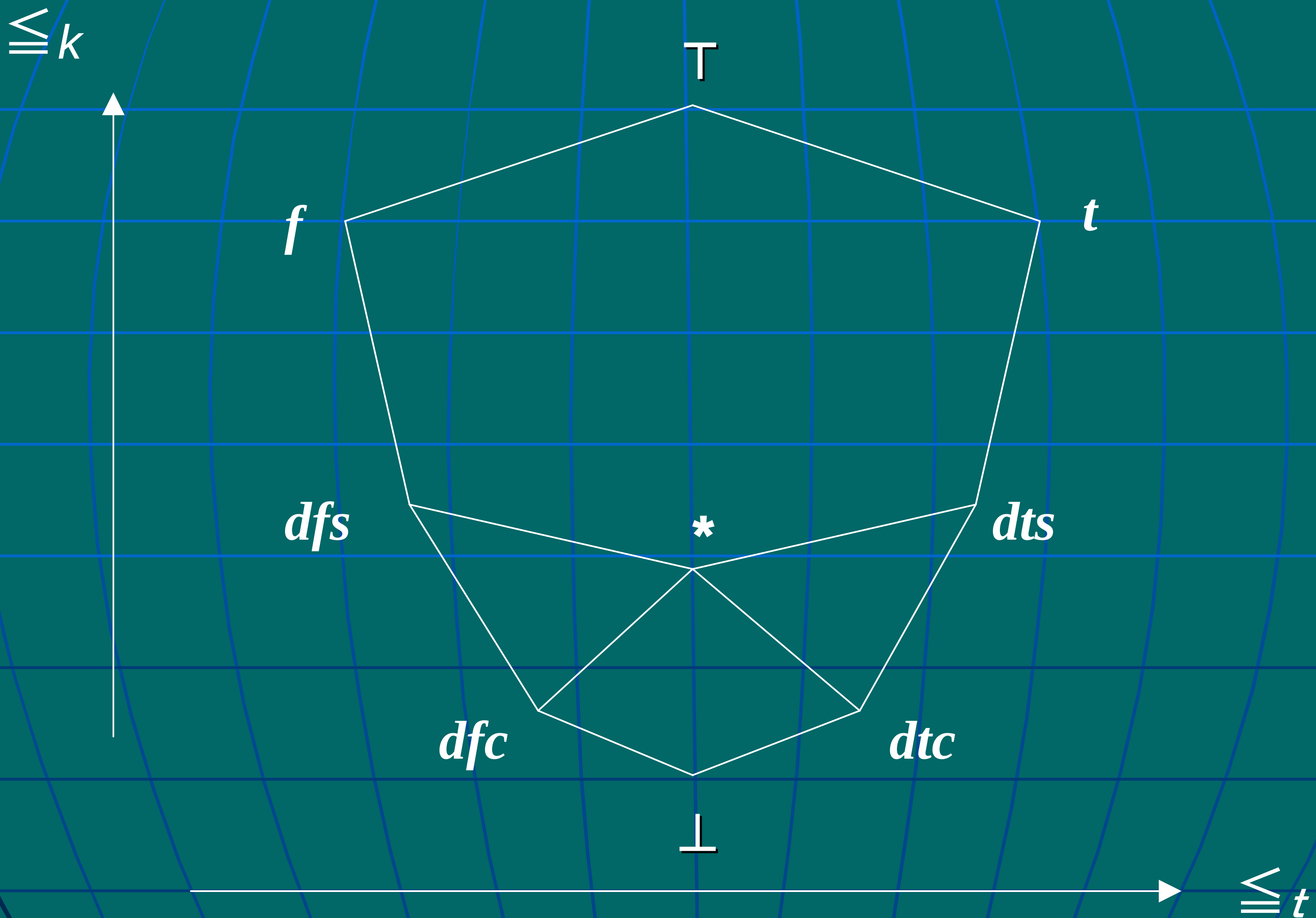
Default Logic

- A default theory is a pair $\Delta = (D,W)$ where D is a set of default rules and W is a set of first-order formulas.
- Given a default theory Δ, a formula is a credulous conclusion if it belongs to some (but not all) extensions of Δ. A formula is a skeptical conclusion if it belongs to all extensions of Δ.

9-valued Logic

- The logic *IX* has the nine truth values *t* (true), *f* (false), ⊤ (contradictory), ⊥ (undefined), ***dts*** (skeptically true by default), ***dfs*** (skeptically false by default), ***dtc*** (credulously true by default), ***dfc*** (credulously false by default), ***** (contradictory by default).

A bilattice for logic *IX*



9-valued Interpretation of Default Theories

Let $\Delta = (D, W)$ be a default theory and $EXT(\Delta)$ the set of all extensions of Δ. Given a propositional formula F, define the mapping ϕ_Δ as

$$\phi_\Delta(F) = \begin{cases} \mathbf{t} & \text{if } EXT(\Delta) \neq \phi \text{ and } W \models F; \\ \mathbf{f} & \text{if } EXT(\Delta) \neq \phi \text{ and } W \models \neg F; \end{cases}$$

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\phi_{\Delta}(F) = \begin{cases}
\textbf{\textit{dts}} & \text{if } EXT(\Delta) \neq \phi,\ W \not\models F \\
& \quad \text{and } \forall E \in EXT(\Delta),\ F \in E; \\
\textbf{\textit{dfs}} & \text{if } EXT(\Delta) \neq \phi,\ W \not\models \neg F \\
& \quad \text{and } \forall E \in EXT(\Delta),\ \neg F \in E; \\
\textbf{\textit{dtc}} & \text{if } \exists E \in EXT(\Delta) \text{ s.t. } F \in E \\
& \quad \text{and } \exists E' \in EXT(\Delta) \text{ s.t. } F \notin E'; \\
\textbf{\textit{dfc}} & \text{if } \exists E \in EXT(\Delta) \text{ s.t. } \neg F \in E \\
& \quad \text{and } \exists E' \in EXT(\Delta) \text{ s.t. } \neg F \notin E'; \\
\bot & \text{otherwise.}
\end{cases}
$$

We write $\phi_{\Delta}(F) = \mathrm{T}$ if $\phi_{\Delta}(F) = \boldsymbol{t}$ and $\phi_{\Delta}(F) = \boldsymbol{f}$;

and $\phi_{\Delta}(F) = *$ if $\phi_{\Delta}(F) = \textbf{\textit{dtc}}$ and $\phi_{\Delta}(F) = \textbf{\textit{dfc}}$.

Example

$$\Delta : \quad bird, \quad \frac{bird : flies}{flies}$$

$$E = Th(\{ bird, \; flies \}).$$

$$\phi_{\Delta}(bird) = \mathbf{t} ,$$

$$\phi_{\Delta}(flies) = \mathbf{dts} ,$$

$$\phi_{\Delta}(bird \rightarrow flies) = \mathbf{dts} , \quad etc$$

Example

$$\Delta : \quad \frac{:\neg rh-broken \wedge lh-broken}{lh-broken}, \quad \frac{:\neg lh-broken \wedge rh-broken}{rh-broken}$$

$E_1 = Th(\{ lh\text{-}broken \})$

$E_2 = Th(\{ rh\text{-}broken \})$

$\phi_{\Delta}(lh\text{-}broken) = \phi_{\Delta}(rh\text{-}broken) =$ **dtc** ,

$\phi_{\Delta}(lh\text{-}broken \vee rh\text{-}broken) =$ **dts** ,

$\phi_{\Delta}(lh\text{-}broken \wedge rh\text{-}broken) = \bot$, *etc*

Example

Δ : *quaker* $\wedge$ *republican*

$$\frac{quaker : pacifist}{pacifist} \quad , \quad \frac{republican : \neg pacifist}{\neg pacifist}$$

$E_1 = Th(\{ quaker \wedge republican,\ pacifist \})$
$E_2 = Th(\{ quaker \wedge republican,\ \neg\, pacifist \})$

$\phi_{\Delta}(quaker \wedge republican) = \mathbf{t}$,
$\phi_{\Delta}(pacifist) = *$, *etc*

Some Properties

- $\phi_\triangle(\neg F) = \neg\phi_\triangle(F)$
- $\phi_\triangle(F) \geqq_k \phi_\triangle(G)$ iff $\phi_\triangle(\neg F) \geqq_k \phi_\triangle(\neg G)$
- $\phi_\triangle(F) \geqq_t \phi_\triangle(G)$ iff $\phi_\triangle(\neg G) \geqq_t \phi_\triangle(\neg F)$
- $\neg(\phi_\triangle(F) \vee \phi_\triangle(G)) = \neg\phi_\triangle(F) \wedge \neg\phi_\triangle(G)$
- $\neg(\phi_\triangle(F) \wedge \phi_\triangle(G)) = \neg\phi_\triangle(F) \vee \neg\phi_\triangle(G)$
- $\phi_\triangle(F \vee G) \geqq_t \phi_\triangle(F) \vee \phi_\triangle(G)$
- $\phi_\triangle(F \wedge G) \leqq_t \phi_\triangle(F) \wedge \phi_\triangle(G)$
- $\phi_\triangle(F \rightarrow G) \geqq_t \phi_\triangle(\neg F) \vee \phi_\triangle(G)$

Order Relation between Default Theories

- Given two default theories Δ_1 and Δ_2, Δ_1 is stronger than Δ_2 (written $\Delta_2 \leqq_{DL} \Delta_1$) if $\phi_{\Delta_2}(F) \leqq_k \phi_{\Delta_1}(F)$ for any formula F in the language.
- We write $\Delta_1 =_{DL} \Delta_2$ if $\Delta_1 \leqq_{DL} \Delta_2$ and $\Delta_2 \leqq_{DL} \Delta_1$.
- Intuitively, $\Delta_2 \leqq_{DL} \Delta_1$ means that Δ_1 has more certain information than Δ_2 .

Reduction to FOL

For two default theories $\Delta_1 = (\phi, W_1)$ and $\Delta_2 = (\phi, W_2)$,

$$\Delta_2 \leqq_{DL} \Delta_1 \text{ iff } W_1 \models W_2.$$

- The relation $\leqq_{DL}$ is a natural extension of the one for first-order theories.

Order Equivalence

Given two default theories Δ_1 and Δ_2, $\Delta_1 =_{DL} \Delta_2$ implies $Ext(\Delta_1)=Ext(\Delta_2)$, but not vice-versa.

- The order-equivalence relation $=_{DL}$ is stronger than the equivalence based on extensions.

Example

Δ_1: *bird*, *bird* $\rightarrow$ *flies*,

Δ_2: *bird*, $\dfrac{bird : flies}{flies}$

$\phi_{\Delta 1}(bird) = \phi_{\Delta 2}(bird) = \mathbf{t}$, $\phi_{\Delta 1}(flies) = \mathbf{t}$,
$\phi_{\Delta 2}(flies) = \mathbf{dts}$. Then, $\Delta_2 \leqq_{DL} \Delta_1$.

Next, given

Δ_3: *bird*, $\dfrac{: bird \rightarrow flies}{bird \rightarrow flies}$,

it holds that $\Delta_2 =_{DL} \Delta_3$.

Nonmonotonicity of $\leqq_{DL}$

- Given two default theories Δ_1 and Δ_2, $\Delta_1 \leqq_{DL} \Delta_2$ implies neither
  $\Delta_1 \leqq_{DL} \Delta_2 \cup \{F\}$ nor
  $\Delta_1 \cup \{F\} \leqq_{DL} \Delta_2 \cup \{F\}$
  for any formula *F*.
- In particular, $\Delta_1 \not\leqq_{DL} \Delta_1 \cup \{F\}$ in general.
- Introduction of formulas does not monotonically increase information.

Connection between $\leqq_{DL}$ and Default Extensions

Let $\Delta_1 = (D_1, W_1)$ and $\Delta_2 = (D_2, W_2)$ be two default theories. Then, $\Delta_1 \leqq_{DL} \Delta_2$ if the following conditions are satisfied:

1. $W_2 \models W_1$,
2. $\forall E_2 \in Ext(\Delta_2)$, $\exists E_1 \in Ext(\Delta_1)$ s.t. $E_1 \subseteq E_2$,
3. $\forall E_1 \in Ext(\Delta_1)$, $\exists E_2 \in Ext(\Delta_2)$ s.t. $E_1 \subseteq E_2$.

Ordering Nonmonotonic Logic Programs

- An extended logic program (ELP) Π is a set of rules of the form:

  $$L_0 \leftarrow L_1, \ldots, L_m, \, not\, L_{m+1}, \ldots, not\, L_n$$

  where L_i is a literal and *not* represents *negation as failure*.
- The semantics of an ELP is given by the answer set semantics .

Connection between DL and ELP [Gelfond & Lifschitz, 1991]

- The rule $L_0 \leftarrow L_1, \ldots, L_m, \mathit{not}\ L_{m+1}, \ldots, \mathit{not}\ L_n$ is interpreted as the default rule

$$\frac{L_1 \wedge \cdots \wedge L_m : \neg L_{m+1}, \ldots, \neg L_n}{L_0}$$

- There is a 1-1 correspondence between an answer set of a program and a default extension of its corresponding default theory.

Ordering ELPs

- The mapping ϕ_{Π} and 9-valued interpretations of ELPs are introduced in a manner similar to DL.
- The ordering $\leqq_{AS}$ is introduced over ELPs.
- A sufficient condition to judge the order relation between two ELPs is given using the answer sets of those programs.

9-valued Interpretation of ELPs

Let Π be an ELP, AS(Π) the set of all answer sets of Π, Π^+ the set of not-free rules from Π, and $Cn(\Pi^+)$ the set of smallest set of ground literals (logically) closed under Π^+. For a positive literal L, define the mapping ϕ_Π as

$$\phi_\Pi(L) = \begin{cases} \mathbf{t} \ \text{ if } AS(\Pi) \neq \phi \ \text{ and } \ L \in Cn(\Pi^+) \ ; \\ \mathbf{f} \ \text{ if } AS(\Pi) \neq \phi \ \text{ and } \neg L \in Cn(\Pi^+) \ ; \end{cases}$$

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\phi_{\Pi}(L) = \begin{cases}
\textbf{dts} & \text{if } AS(\Pi) \neq \phi,\ L \notin Cn(\Pi^{+}) \\
& \quad \text{and } \forall S \in AS(\Pi),\ L \in S\ ; \\
\textbf{dfs} & \text{if } AS(\Pi) \neq \phi,\ \neg L \notin Cn(\Pi^{+}) \\
& \quad \text{and } \forall S \in AS(\Pi),\ \neg L \in S\ ; \\
\textbf{dtc} & \text{if } \exists S \in AS(\Pi) \text{ s.t. } L \in S \\
& \quad \text{and } \exists T \in AS(\Pi) \text{ s.t. } L \notin T\ ; \\
\textbf{dfc} & \text{if } \exists S \in AS(\Pi) \text{ s.t. } \neg L \in S \\
& \quad \text{and } \exists T \in AS(\Pi) \text{ s.t. } \neg L \notin T\ ; \\
\bot & \text{otherwise.}
\end{cases}
$$

We write $\phi_{\Pi}(L) = T$ if $\phi_{\Pi}(L) = t$ and $\phi_{\Pi}(L) = f$;
and $\phi_{\Pi}(L) = *$ if $\phi_{\Pi}(L) = dtc$ and $\phi_{\Pi}(L) = dfc$.

Order Relation between ELPs

- Given two ELPs Π_1 and Π_2, Π_1 is stronger than Π_2 under the answer set semantics (written $\Pi_2 \leqq_{AS} \Pi_1$) if $\phi_{\Pi_2}(L) \leqq_k \phi_{\Pi_1}(L)$ for any literal L in the language.
- We write $\Pi_1 =_{AS} \Pi_2$ if $\Pi_1 \leqq_{AS} \Pi_2$ and $\Pi_2 \leqq_{AS} \Pi_1$.

Some Properties

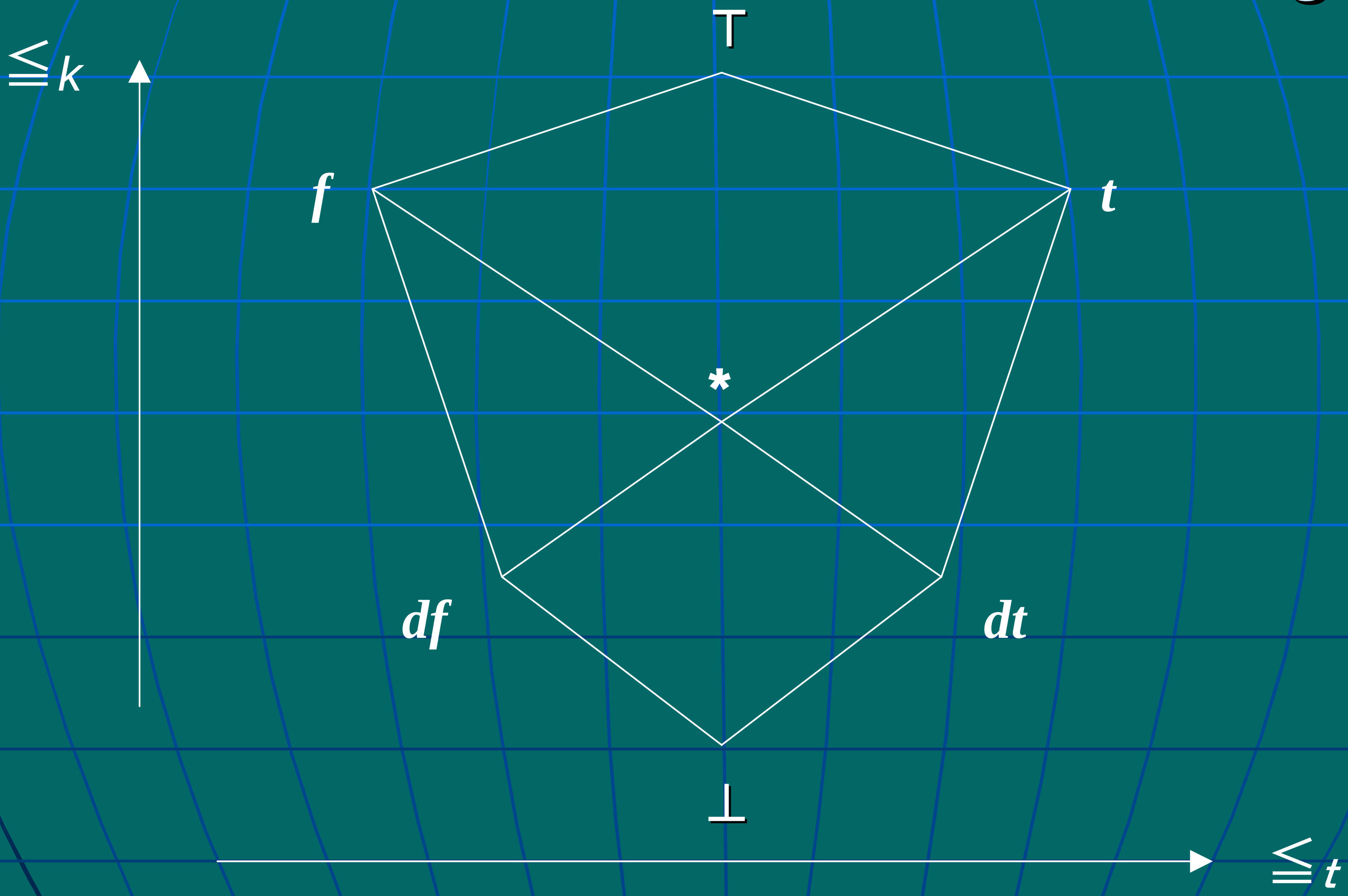
- For two not-free ELPs Π_1 and Π_2, $\Pi_1 \leqq_{AS} \Pi_2$ iff $Cn(\Pi_1) \subseteq Cn(\Pi_2)$.
- Given two ELPs Π_1 and Π_2, $\Pi_1 =_{AS} \Pi_2$ implies $AS(\Pi_1)=AS(\Pi_2)$, but not vice-versa.
- $\leqq_{AS}$ is nonmonotonic wrt the introduction of new rules to a program.

Connection between $\leqq_{AS}$ and Answer Sets

Let Π_1 and Π_2 be two ELPs. Then, $\Pi_1 \leqq_{AS} \Pi_2$ if the following conditions are satisfied:

1. $Cn(\Pi_1^+) \subseteq Cn(\Pi_2^+)$,
2. $\forall S \in AS(\Pi_2), \exists T \in AS(\Pi_1)$ s.t. $T \subseteq S$,
3. $\forall T \in AS(\Pi_1), \exists S \in AS(\Pi_2)$ s.t. $T \subseteq S$.

Comparison with Ginsberg's 7-valued bilattice for default logic



Comparison with Ginsberg's Logic

Δ :

$$\frac{: p \wedge q}{p \wedge q} \;, \qquad \frac{: \neg p}{\neg p}$$

$$E_1 = Th(\{ p \wedge q \}) \;, \quad E_2 = Th(\{ \neg p \})$$

$$\phi_{\Delta}(p) = * \;, \; \phi_{\Delta}(q) = \boldsymbol{dtc} \;, \; \phi_{\Delta}(\neg p \vee q) = \boldsymbol{dts}.$$

! Ginsberg interprets p as $*$, but handles both q and $\neg p \vee q$ as $\boldsymbol{dt}$.

Computational Aspect

- There is a difficulty for computing $\phi_{\Delta}(F)$ for an arbitrary formula *F*.
- An order relation $\Delta_1 \leqq_{DL} \Delta_2$ is judged using default extensions.
- In logic programming, the relation $\Pi_1 \leqq_{AS} \Pi_2$ is checked using proof procedures for answer set programming.

Application to Inductive Logic Programming

The relation $\Pi_2 \leqq_{AS} \Pi_1$ holds for

$$\Pi_1 = \{\ flies(x) \leftarrow bird(x),\ bird(tweety) \leftarrow\ \},$$

$$\Pi_2 = \{\ flies(x) \leftarrow bird(x),\ not\ ab(x),\ bird(tweety) \leftarrow\ \}.$$

If we read the relation $\leqq_{AS}$ as "more general", the ordering gives a theoretical ground for inductive generalization in nonmonotonic ILP.

Conclusion

- Multi-valued interpretation of default logic is introduced, that can distinguish definite and skeptical/credulous default conclusions.
- Based on this, an ordering for default theories is introduced and applied to nonmonotonic logic programming.
- The ordering has potential application to the theory of induction in NML.

Final Remarks

- The results provide a method of comparing default theories or logic programs in a manner different from the conventional extension-based or model-based standpoint.
- The 9-valued bilattice could be used for characterizing other NMR formalisms having the same inference modes as DL.