Ordering Argumentation Frameworks

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Comparing Different Theories

Compare the amount of information included in two different theories (or knowledge bases).

- **Which theory is more informative?**
- **How to assess the amount of information in a theory?**
- **Under what condition two theories are considered to be equivalent?**

Comparing First-Order Theories

Given two first-order theories T_1 and T_2,

- T_1 is **more general** (or **informative**) than T_2 if $T_1 \models T_2$.
- T_1 is **equivalent** to T_2 ($T_1 \equiv T_2$) if $T_1 \models T_2$ and $T_2 \models T_1$.
- E.g. $p \models p \vee q$ (p is more informative than $p \vee q$)

Comparing Nonmonotonic Theories

Given two logic programs P_1 and P_2,

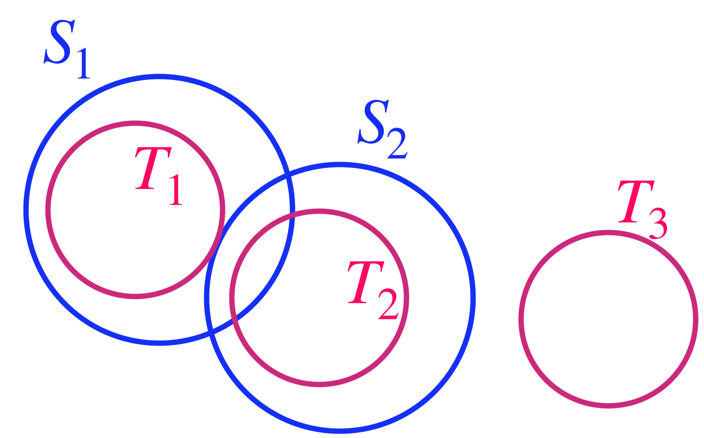
$P_1 = \{\, p \leftarrow not\, q \,\}$ **answer set**: $\{p\}$

$P_2 = \{\, p \leftarrow not\, q, \quad q \leftarrow not\, p \,\}$ **answer sets**: $\{p\}, \{q\}$.

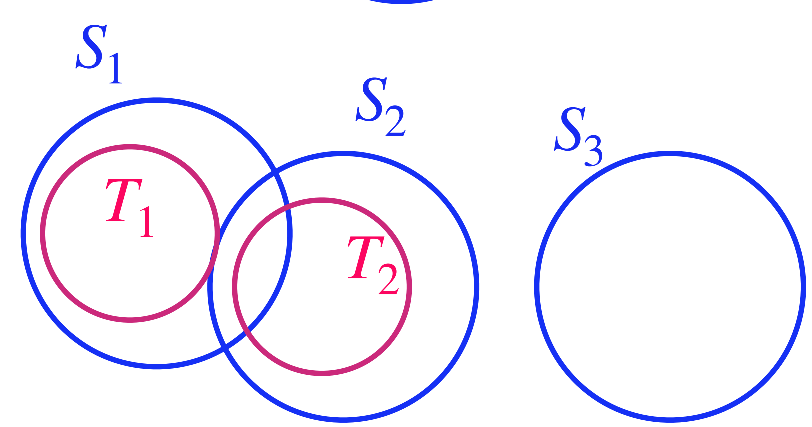
- P_1 is **more informative** than P_2 because p is inferred by **skeptical reasoning** in P_1 (but not in P_2).
- P_2 is **more informative** than P_1 because q is inferred by **credulous reasoning** in P_2 (but not in P_1).

Ordering Logic Programs
(Inoue & Sakama, ICLP 2006)

Given two logic programs P_1 and P_2,



- $P_1 \models^{\sharp} P_2$ (P_1 is **more ♯-general** than P_2) iff for any answer set S of P_1 there is an answer set T of P_2 s.t. $T \subseteq S$.



- $P_1 \models^{\flat} P_2$ (P_1 is **more ♭-general** than P_2) iff for any answer set T of P_2 there is an answer set S of P_1 s.t. $T \subseteq S$.

- If $P_1 \models^{\sharp} P_2$ (resp. $P_1 \models^{\flat} P_2$) then P_1 entails more **skeptical** (resp. **credulous**) consequences than P_2 under the answer set semantics.

Purposes

Compare acceptable arguments in **abstract argumentation frameworks** (AFs).

- **Which AF is more skeptical/credulous in reasoning about arguments?**
- **Does the result change if different argumentation semantics is used?**
- **Is there any connection to equivalence of AFs?**

Argumentation Framework

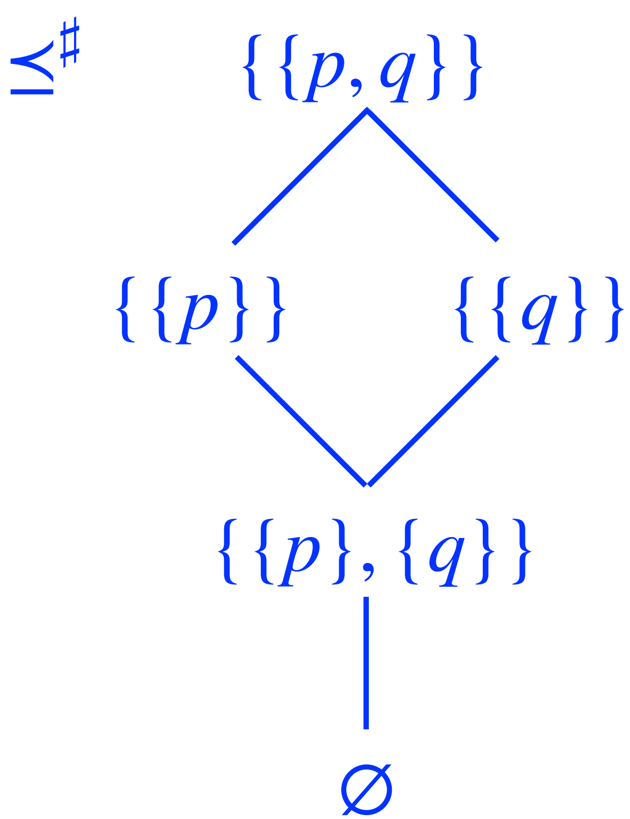
- An **argumentation framework** (AF) is a pair (A, R) where A is a finite set of **arguments** and $R \subseteq A \times A$ is the **attack relation**.
- We consider **admissible sets** ($\mathcal{E}_{AF}^{adm}$), **complete extensions** ($\mathcal{E}_{AF}^{com}$), **stable extensions** ($\mathcal{E}_{AF}^{stb}$), **preferred extensions** ($\mathcal{E}_{AF}^{prf}$), and the **grounded extension** ($\mathcal{E}_{AF}^{grd}$) of an AF by (Dung, 1995).
- An argument $a \in A$ is **credulously** (resp. **skeptically**) **accepted** under the σ semantics of AF iff $a \in E$ for some (resp. every) $E \in \mathcal{E}_{AF}^{\sigma}$ where $\sigma \in \{adm, com, stb, prf, grd\}$.
- The set of all credulously (resp. skeptically) accepted arguments under the σ semantics of AF is denoted by $crd^{\sigma}(AF)$ (resp. $skp^{\sigma}(AF)$).

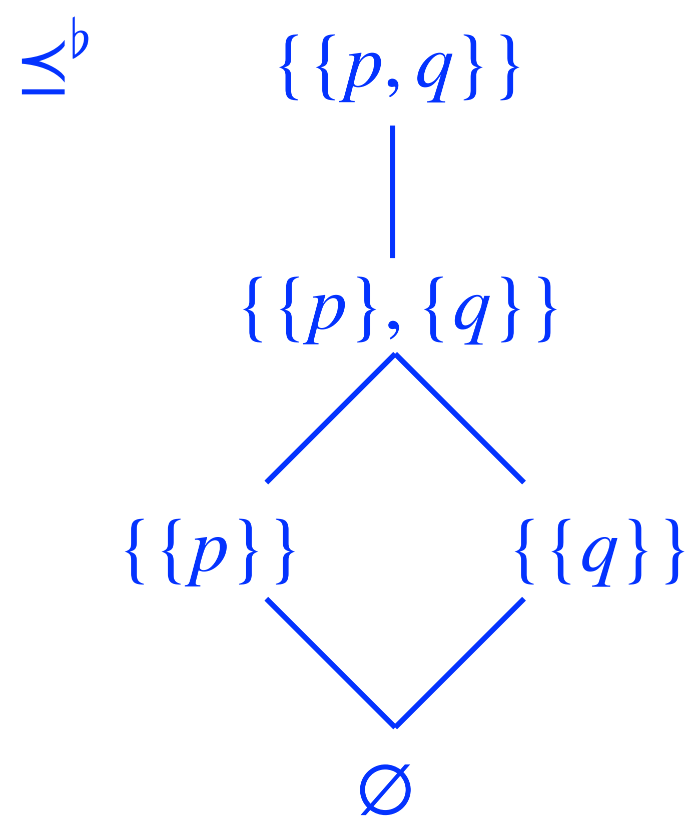
Ordering on Powersets

- D : a set, $\mathscr{P}(D)$: the power set of D.
- Given a partially ordered set $\langle D, \preccurlyeq \rangle$ and $X, Y \in \mathscr{P}(D)$,
  - $X \preceq^{\sharp} Y$ iff $\forall y \in Y \, \exists x \in X$ s.t. $x \preccurlyeq y$ (**Smyth order**)
  - $X \preceq^{\flat} Y$ iff $\forall x \in X \, \exists y \in Y$ s.t. $x \preccurlyeq y$ (**Hoare order**)
- Both $\langle \mathscr{P}(D), \preceq^{\sharp} \rangle$ and $\langle \mathscr{P}(D), \preceq^{\flat} \rangle$ are pre-ordered sets.

Smyth vs. Hoare Order

- $\emptyset \preceq^{\sharp} \{\{p\},\{q\}\} \preceq^{\sharp} \{\{p\}\}, \{\{q\}\} \preceq^{\sharp} \{\{p,q\}\}$ **(Smyth)**
- $\emptyset \preceq^{\flat} \{\{p\}\}, \{\{q\}\} \preceq^{\flat} \{\{p\},\{q\}\} \preceq^{\flat} \{\{p,q\}\}$ **(Hoare)**





Ordering AFs

- $\mathscr{U}$: the universe of all arguments, $\langle \mathscr{P}(\mathscr{U}), \subseteq \rangle$: a poset
- For any Σ_1 and Σ_2 in $\mathscr{P}(\mathscr{P}(\mathscr{U}))$,
    - $\Sigma_1 \preceq^\sharp \Sigma_2$ iff $\forall T \in \Sigma_2 \exists S \in \Sigma_1$ s.t. $S \subseteq T$
    - $\Sigma_1 \preceq^\flat \Sigma_2$ iff $\forall S \in \Sigma_1 \exists T \in \Sigma_2$ s.t. $S \subseteq T$
- Let AF_1 and AF_2 be two AFs, and $\sigma \in \{adm, com, stb, prf, grd\}$.
    - $AF_1 \sqsubseteq_\sigma^\sharp AF_2$ iff $\mathscr{E}_{AF_1}^\sigma \preceq^\sharp \mathscr{E}_{AF_2}^\sigma$
    - $AF_1 \sqsubseteq_\sigma^\flat AF_2$ iff $\mathscr{E}_{AF_1}^\sigma \preceq^\flat \mathscr{E}_{AF_2}^\sigma$

Ordering AFs

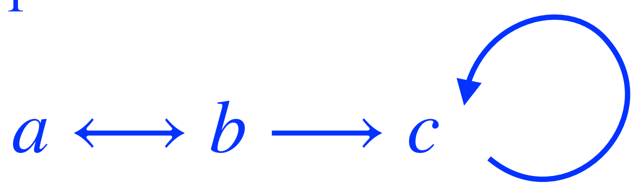
- AF_2 is **more** (or **equally**) $\sharp$-**general** (resp. $\flat$-**general**) than AF_1 (under σ-semantics) if $AF_1 \sqsubseteq^{\sharp}_{\sigma} AF_2$ (resp. $AF_1 \sqsubseteq^{\flat}_{\sigma} AF_2$).

- We write $AF_1 \equiv^{\sharp}_{\sigma} AF_2$ (resp. $AF_1 \equiv^{\flat}_{\sigma} AF_2$) iff $AF_1 \sqsubseteq^{\sharp}_{\sigma} AF_2 \sqsubseteq^{\sharp}_{\sigma} AF_1$ (resp. $AF_1 \sqsubseteq^{\flat}_{\sigma} AF_2 \sqsubseteq^{\flat}_{\sigma} AF_1$).

- $\mathscr{AF}$: the collection of all AFs induced by $\mathscr{U}$. Then $\langle \mathscr{AF}, \sqsubseteq^{\sharp/\flat}_{\sigma} \rangle$ is a pre-ordered set.

Example

AF_1:



AF_2:

$$a \longleftrightarrow b \longleftrightarrow c$$

$$\mathcal{E}^{com}_{AF_1} = \{\emptyset, \{a\}, \{b\}\},\ \mathcal{E}^{prf}_{AF_1} = \{\{a\}, \{b\}\},\ \mathcal{E}^{stb}_{AF_1} = \{\{b\}\},\ \mathcal{E}^{grd}_{AF_1} = \{\emptyset\}.$$

$$\mathcal{E}^{com}_{AF_2} = \{\emptyset, \{a, c\}, \{b\}\},\ \mathcal{E}^{prf}_{AF_2} = \mathcal{E}^{stb}_{AF_2} = \{\{a, c\}, \{b\}\},\ \mathcal{E}^{grd}_{AF_2} = \{\emptyset\}.$$

- $AF_1 \sqsubseteq^{\sharp}_{\sigma} AF_2$ holds for $\sigma \in \{com, prf, grd\}$.
- $AF_1 \sqsubseteq^{\flat}_{\sigma} AF_2$ holds for $\sigma \in \{com, prf, stb, grd\}$.
- $AF_1 \equiv^{\sharp/\flat}_{\sigma} AF_2$ holds for $\sigma = grd$.

Formal Properties

AF_1, AF_2 : two AFs, and $\sigma \in \{adm, com, stb, prf, grd\}$.

- $AF_1 \sqsubseteq^{\sharp}_{adm} AF_2$
- $AF_1 \sqsubseteq^{\sharp}_{grd} AF_2$ iff $AF_1 \sqsubseteq^{\flat}_{grd} AF_2$.
- If $\mathscr{E}^{\sigma}_{AF_1} \subseteq \mathscr{E}^{\sigma}_{AF_2}$ then $AF_1 \sqsubseteq^{\flat}_{\sigma} AF_2$ and $AF_2 \sqsubseteq^{\sharp}_{\sigma} AF_1$.
- $AF_1 \equiv^{\sharp}_{\sigma} AF_2$ iff $min_{\subseteq}(\mathscr{E}^{\sigma}_{AF_1}) = min_{\subseteq}(\mathscr{E}^{\sigma}_{AF_2})$.
- $AF_1 \equiv^{\flat}_{\sigma} AF_2$ iff $max_{\subseteq}(\mathscr{E}^{\sigma}_{AF_1}) = max_{\subseteq}(\mathscr{E}^{\sigma}_{AF_2})$.

Equivalence

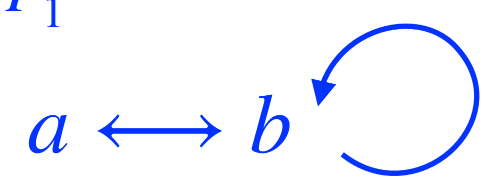
- For $\sigma \in \{stb, prf, grd\}$, it holds that

  $AF_1 \equiv^{\sharp}_{\sigma} AF_2$ iff $AF_1 \equiv^{\flat}_{\sigma} AF_2$ iff $AF_1 \equiv_{\sigma} AF_2$

  where $AF_1 \equiv_{\sigma} AF_2$ iff $\mathcal{E}^{\sigma}_{AF_1} \equiv \mathcal{E}^{\sigma}_{AF_2}$.

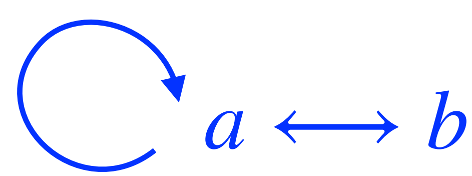
- The result does not hold for any semantics that does not satisfy the **anti-chain** property.

AF_1



$\mathcal{E}^{com}_{AF_1} = \{\emptyset, \{a\}\}$

AF_2



$\mathcal{E}^{com}_{AF_2} = \{\emptyset, \{b\}\}$

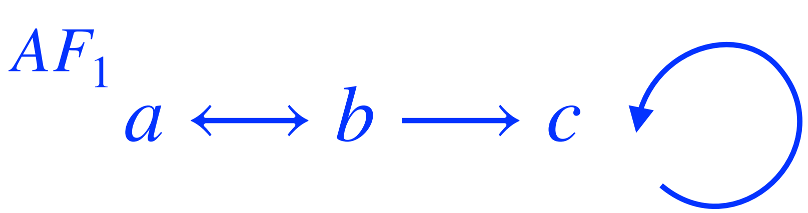
$AF_1 \equiv^{\sharp}_{com} AF_2$

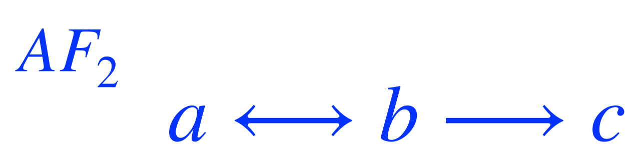
$AF_1 \not\equiv_{com} AF_2$

Skeptical vs. Credulous

For $\sigma \in \{adm, com, stb, prf, grd\}$, it holds that

- If $AF_1 \sqsubseteq^{\flat}_{\sigma} AF_2$ then $crd^{\sigma}(AF_1) \subseteq crd^{\sigma}(AF_2)$.
- If $AF_1 \sqsubseteq^{\sharp}_{\sigma} AF_2$ then $skp^{\sigma}(AF_1) \subseteq skp^{\sigma}(AF_2)$.





$\mathcal{E}^{stb}_{AF_1} = \{\{b\}\}$

$\mathcal{E}^{stb}_{AF_2} = \{\{a, c\}, \{b\}\}$

$AF_1 \sqsubseteq^{\flat}_{stb} AF_2$ implies $crd^{stb}(AF_1) \subseteq crd^{stb}(AF_2)$

$AF_2 \sqsubseteq^{\sharp}_{stb} AF_1$ implies $skp^{stb}(AF_2) \subseteq skp^{stb}(AF_1)$

Comparing Different Semantics

- $\mathcal{E}^{stb}_{AF} \preceq^{\flat} \mathcal{E}^{prf}_{AF} \preceq^{\flat} \mathcal{E}^{com}_{AF} \preceq^{\flat} \mathcal{E}^{adm}_{AF}$
- $\mathcal{E}^{grd}_{AF} \preceq^{\flat} \mathcal{E}^{com}_{AF}$
- $\mathcal{E}^{\sigma}_{AF} \preceq^{\flat} \mathcal{E}^{com}_{AF}$ for $\sigma \in \{adm, com, stb, prf, grd\}$
- $\mathcal{E}^{adm}_{AF} \preceq^{\sharp} \mathcal{E}^{com}_{AF} \preceq^{\sharp} \mathcal{E}^{prf}_{AF} \preceq^{\sharp} \mathcal{E}^{stb}_{AF}$
- $\mathcal{E}^{com}_{AF} \preceq^{\sharp} \mathcal{E}^{grd}_{AF}$
- $\mathcal{E}^{grd}_{AF} \preceq^{\sharp} \mathcal{E}^{\lambda}_{AF}$ for $\lambda \in \{com, stb, prf, grd\}$

Minimal Upper & Maximal Lower Bounds

Σ_1, Σ_2 : two antichain sets in $\langle \mathscr{P}(\mathscr{U}), \subseteq \rangle$

- $\Sigma \in \mathscr{P}(\mathscr{P}(\mathscr{U}))$ is an **mub** of Σ_1 and Σ_2 in $\langle \mathscr{P}(\mathscr{P}(\mathscr{U})), \preceq^\sharp \rangle$ iff $\Sigma = min_\subseteq(X)$ where $X = \{S \cup T \mid S \in \Sigma_1$ and $T \in \Sigma_2\}$.
- $\Sigma \in \mathscr{P}(\mathscr{P}(\mathscr{U}))$ is an **mlb** of Σ_1 and Σ_2 in $\langle \mathscr{P}(\mathscr{P}(\mathscr{U})), \preceq^\sharp \rangle$ iff $\Sigma = min_\subseteq(\Sigma_1 \cup \Sigma_2)$.
- $\Sigma \in \mathscr{P}(\mathscr{P}(\mathscr{U}))$ is an **mub** of Σ_1 and Σ_2 in $\langle \mathscr{P}(\mathscr{P}(\mathscr{U})), \preceq^\flat \rangle$ iff $\Sigma = max_\subseteq(\Sigma_1 \cup \Sigma_2)$.
- $\Sigma \in \mathscr{P}(\mathscr{P}(\mathscr{U}))$ is an **mlb** of Σ_1 and Σ_2 in $\langle \mathscr{P}(\mathscr{P}(\mathscr{U})), \preceq^\flat \rangle$ iff $\Sigma = max_\subseteq(Y)$ where $Y = \{S \cap T \mid S \in \Sigma_1$ and $T \in \Sigma_2\}$.

Does mub or mlb of two AFs always exist?

- Given AF_1, AF_2 , is there AF s.t. $\mathscr{E}^{\sigma}_{AF}$ is obtained as an mub (or mlb) of $\mathscr{E}^{\sigma}_{AF_1}$ and $\mathscr{E}^{\sigma}_{AF_2}$?
- Consider AF_1 and AF_2 s.t. $\mathscr{E}^{stb}_{AF_1} = \{\{a, b\}, \{a, c\}\}$ and $\mathscr{E}^{stb}_{AF_2} = \{\{b, c\}\}$. Then, $min_{\subseteq}(\mathscr{E}^{stb}_{AF_1} \cup \mathscr{E}^{stb}_{AF_2}) = max_{\subseteq}(\mathscr{E}^{stb}_{AF_1} \cup \mathscr{E}^{stb}_{AF_2})$ $= \{\{a, b\}, \{a, c\}, \{b, c\}\}$.
- But there is no AF s.t. $\mathscr{E}^{stb}_{AF} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$, since any stable extension must be **tight** and $\mathscr{E}^{stb}_{AF}$ does not satisfy this condition.

Strong Ordering

- Compare two AFs under dynamic environments by observing the effect of incorporating new info.
- Given $AF_1 = (A_1, R_1)$, $AF_2 = (A_2, R_2)$, define

  $AF_1 \sqcup AF_2 = (A_1 \cup A_2, R_1 \cup R_2)$.
- $AF_1 \trianglelefteq^{\sharp}_{\sigma} AF_2$ iff $(AF_1 \sqcup AF) \sqsubseteq^{\sharp}_{\sigma} (AF_2 \sqcup AF)$ for any $AF \in \mathscr{AF}$
- $AF_1 \trianglelefteq^{\flat}_{\sigma} AF_2$ iff $(AF_1 \sqcup AF) \sqsubseteq^{\flat}_{\sigma} (AF_2 \sqcup AF)$ for any $AF \in \mathscr{AF}$

  where $\sigma \in \{adm, com, stb, prf, grd\}$.

Connection to Strong Equivalence

- AF_1 and AF_2 are **strongly equivalent** (wrt σ semantics) if $AF_1 \sqcup AF \equiv_\sigma AF_2 \sqcup AF$ for any $AF \in \mathscr{AF}$.

- The following result holds for $\sigma \in \{prf, stb, grd\}$:

$$AF_1 \trianglelefteq^\sharp_\sigma AF_2 \trianglelefteq^\sharp_\sigma AF_1 \quad \text{iff} \quad AF_1 \trianglelefteq^\flat_\sigma AF_2 \trianglelefteq^\flat_\sigma AF_1$$

iff AF_1 and AF_2 are strongly equivalent.

Strong Ordering vs. Extension Inclusion

The following results hold for $\sigma \in \{stb, prf, grd\}$:

- if $AF_1 \trianglelefteq^{\flat}_{\sigma} AF_2$ then $\mathscr{E}^{\sigma}_{AF_1} \subseteq \mathscr{E}^{\sigma}_{AF_2}$.
- if $AF_1 \trianglelefteq^{\sharp}_{\sigma} AF_2$ then $\mathscr{E}^{\sigma}_{AF_2} \subseteq \mathscr{E}^{\sigma}_{AF_1}$.
- $AF_1 \trianglelefteq^{\flat}_{\sigma} AF_2$ iff $AF_2 \trianglelefteq^{\sharp}_{\sigma} AF_1$
  iff $\mathscr{E}^{\sigma}_{AF_1 \sqcup AF} \subseteq \mathscr{E}^{\sigma}_{AF_2 \sqcup AF}$ for any $AF \in \mathscr{AF}$.

Conclusion

- Two orderings $\sqsubseteq^{\sharp}_{\sigma}$ and $\sqsubseteq^{\flat}_{\sigma}$ are used for comparing skeptical/credulous acceptance of arguments in different argumentation frameworks.
- Two orderings $\trianglelefteq^{\sharp}_{\sigma}$ and $\trianglelefteq^{\flat}_{\sigma}$ have connection to strong equivalence of AFs.
- The results are independent of particular semantics and applied to other semantics as well.