Generality and Equivalence Relations in Default Logic

Katsumi Inoue
National Institute of Informatics

Chiaki Sakama
Wakayama University

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Comparing the Amounts of Information between Programs

- ⌘ Assessment of relative value of each theory/ontology
  - ⌘ **Generality**/Specificity and Abstraction/Refinement
  - ⌘ **Equivalence**/Non-equivalence
  - ⌘ **Strength**/Weakness and Higher/Lower Priority
- ⌘ Theory of generality is central in **Inductive Logic Programming** (ILP), in which domain-independent criteria to compose better theories are investigated.
- ⌘ Synthesizing a common generalized/specialized theory from different sources of information is important in **Multi-Agent Systems** (MAS).

Comparing First-order Theories

- T1, T2: first-order theory
- T1 is **more general** (or stronger) **than** T2 if T1 $\models$ T2 [Plotkin; Niblett].
- e.g., $\{ p, \; p \rightarrow q \}$ is stronger than $\{ p \vee q \}$.
- T1 and T2 are **logically equivalent** if T1$\equiv$T2, i.e., T1 $\models$ T2 and T2 $\models$ T1.
- Logically equivalent theories belong to the same equivalence class of the generality relation.

Comparing Nonmonotonic Theories

- T1, T2: (nonmonotonic) theories
- When can we say that T1 is **more general** than (or is **more informative** than) T2?
- T1 and T2 are **equivalent** if T1 and T2 have the same semantical meaning:
  - *weak/strong equivalence* [Lifschitz et al., Turner]
- Under which generality relation do equivalent theories belong to the same equivalence class?

Comparing Nonmonotonic Theories

⌘ **Example**:

$$\Delta 1: \quad \frac{:p}{p}$$

$$\Delta 2: \quad \frac{:\neg q}{p}, \quad \frac{:\neg p}{q}$$

$\Delta 1$ has the extension: cl({p})
$\Delta 2$ has the extensions: cl({p}), cl({q})

⌘ $\Delta 1$ is *more informative* than $\Delta 2$ in the sense that $\Delta 1$ has the **skeptical** consequences cl({p}) $\supseteq$ cl({$p \vee q$}).

⌘ $\Delta 2$ is *more informative* than $\Delta 1$ in the sense that $\Delta 2$ has the **credulous** consequences cl({p}) $\cup$ cl({q}) $\supseteq$ cl({p}).

😐 Thus, several generality measures can be considered.

Goals

- We construct multiple criteria to decide if a theory is more general than another theory in **(disjunctive) default logic**.
- Generality relations are mathematically defined as **pre-orders** based on **comparing sets of extensions**.
  - Any pair of theories should have both **minimal upper and maximal lower bounds** under such generality orderings.
  - We devise those generality orderings in such a way that any pair of **equivalent theories belong to the same equivalence class** that is induced from such pre-ordered sets.
- We also provide the notion of **strong generality** that implies strong equivalence within the same equivalence class.
- These generality relations should **extend** both those for **first-order theories** [Niblett] and those for **answer set programming** [Inoue & Sakama, ICLP'06].

Disjunctive Default Theory

⌘ (disjunctive) default ***d***:

$$\frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma_1 \mid \cdots \mid \gamma_n}$$

where $\alpha, \beta_1, \ldots \beta_m, \ldots, \gamma_1, \ldots, \gamma_n$ are propositional formulas.

preq(***d***)=$\{\alpha\}$, just(***d***)=$\{\beta_1, \ldots, \beta_m\}$, cons(***d***)=$\{\gamma_1, \cdots, \gamma_n\}$.

⌘ Reiter's (non-disjunctive) default: |cons(***d***)|=1.

⌘ non-default rule: just(***d***)={}.

⌘ (disjunctive) fact: α=**true** & just(***d***)={},

written as $\gamma_1 \mid \cdots \mid \gamma_n$.

Extensions of Default Theories

⌘ Δ : default theory, E : set of formulas

$$\Delta^E = \left\{ \frac{\alpha :}{\gamma_1 \mid \cdots \mid \gamma_n} \;\middle|\; \frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma_1 \mid \cdots \mid \gamma_n} \in \Delta,\; \neg\beta_1, \ldots, \neg\beta_m \notin E \right\}$$

⌘ A set E' is **closed** under the rules of Δ^E if for any default

$$\frac{\alpha :}{\gamma_1 \mid \cdots \mid \gamma_n} \in \Delta^E, \quad \alpha \in E' \text{ implies } \{\gamma_1, \ldots, \gamma_n\} \cap E' \neq \phi .$$

⌘ E is an **extension** of Δ iff E is a minimal set closed under provability in propositional logic and the rules from Δ^E.

⌘ Δ is [**consistent** / **contradictory** / **incoherent**] if it has [a consistent / an inconsistent / no] extension.

Equivalence between Default Theories

- $E(\Delta)$: the set of all extensions of Δ .
- A formula ψ is a **skeptical / credulous consequence** of Δ if ψ belongs to all/some extensions in $E(\Delta)$.
  - $skp(\Delta)$: the set of skeptical consequences of Δ
  - $crd(\Delta)$: the set of credulous consequences of Δ

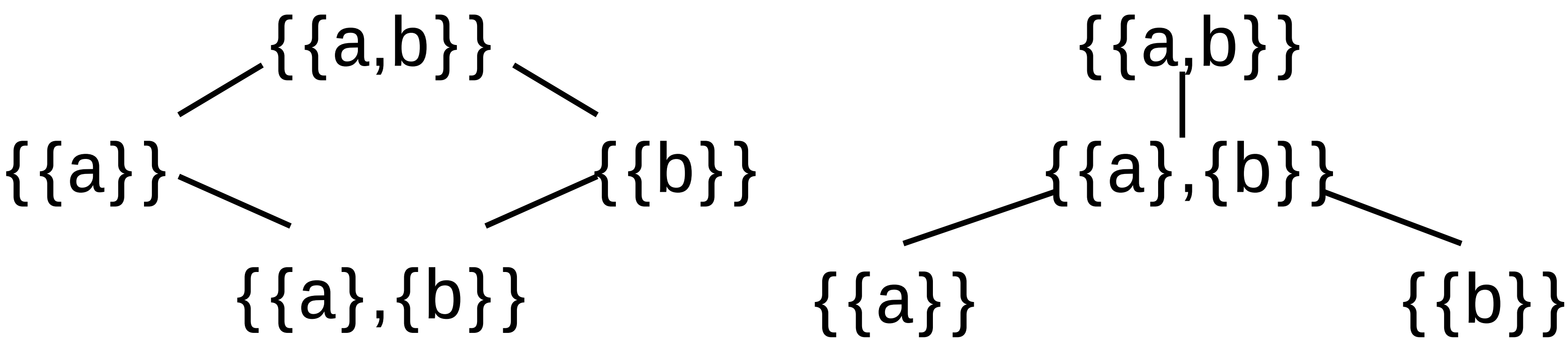
⌘ Δ_1 and Δ_2 are **(weakly) equivalent** if $E(\Delta_1) = E(\Delta_2)$.

⌘ Δ_1 and Δ_2 are **strongly equivalent** [Turner, LPNMR'01] if $E(\Delta_1 \cup \Pi) = E(\Delta_2 \cup \Pi)$ for any default theory Π.

⌘ Strong equivalence implies weak equivalence.

Ordering Extensions: Basic Intuition

- In the FO case, { a $\wedge$ b } is more informative than { a }, which is more informative than { a $\vee$ b }.
  In fact, a $\wedge$ b $\models$ a $\models$ a $\vee$ b.
- In analogy, {{a,b}} $\geq$ {{a}} $\geq$ {{a},{b}}.



Smyth *Hoare*

Ordering on Powersets

⌘ *pre-order* $\leq$: binary relation which is reflexive and transitive

⌘ *partial order* $\leq$: pre-order which is also anti-symmetric

⌘ $\langle D, \leq \rangle$: *pre-ordered set / poset*

⌘ $\wp(D)$: the power set of D

⌘ The **Smyth order**: for $X, Y \in \wp(D)$,

$$X \models^{\#} Y \text{ iff } \forall x \in X\, \exists y \in Y.\ y \leq x$$

⌘ The **Hoare order**: for $X, Y \in \wp(D)$,

$$X \models^{\flat} Y \text{ iff } \forall y \in Y\, \exists x \in X.\ y \leq x$$

⌘ Both $\langle \wp(D), \models^{\#} \rangle$ and $\langle \wp(D), \models^{\flat} \rangle$ are pre-ordered sets.

Ordering Default Theories

⌘ $\langle \mathcal{S}(\mathcal{O}), \subseteq \rangle$: poset

⌘ $\mathcal{DT}$: the class of all default theories

⌘ $\Delta_1, \Delta_2 \in \mathcal{DT}$

- Δ_1 is **more #-general** than Δ_2 :

$$\Delta_1 \models^{\#} \Delta_2 \quad \text{iff} \quad E(\Delta_1) \models^{\#} E(\Delta_2)$$

- Δ_1 is **more ♭-general** than Δ_2 :

$$\Delta_1 \models^{\flat} \Delta_2 \quad \text{iff} \quad E(\Delta_1) \models^{\flat} E(\Delta_2)$$

⌘**Theorem**: $\Delta_1 \models^{\#} \Delta_2$ and $\Delta_2 \models^{\#} \Delta_1$

iff $\Delta_1 \models^{\flat} \Delta_2$ and $\Delta_2 \models^{\flat} \Delta_1$

iff Δ_1 and Δ_2 are weakly equivalent.

Generality Ordering: Example

$$\Delta 1: \quad \frac{:\neg q}{p}$$

$$\Delta 2: \quad \frac{:\neg q}{p}, \quad \frac{:\neg p}{q}$$

$$\Delta 3: \quad p \mid q$$

$$\Delta 4: \quad \frac{:p}{p}, \quad \frac{p:}{q}$$

$E(\Delta 1)=\{\mathrm{cl}(\{p\})\}$, $E(\Delta 2)=E(\Delta 3)=\{\mathrm{cl}(\{p\}), \mathrm{cl}(\{q\})\}$,

$E(\Delta 4)=\{\mathrm{cl}(\{p,q\})\}$

- $\Delta 4 \models^{\sharp} \Delta 1 \models^{\sharp} \Delta 2$
- $\Delta 4 \models^{\flat} \Delta 2 \models^{\flat} \Delta 1$
- $\Delta 2 \models^{\sharp} \Delta 3 \models^{\sharp} \Delta 2$, $\quad \Delta 2 \models^{\flat} \Delta 3 \models^{\flat} \Delta 2$

Minimal Upper / Maximal Lower Bounds

- ⌘ Γ is an <u>*upper bound*</u> of $\Delta 1$ and $\Delta 2$ in $\langle \mathcal{DT}, \models^{\#/\flat} \rangle$ if $\Gamma \models^{\#/\flat} \Delta 1$ and $\Gamma \models^{\#/\flat} \Delta 2$.
- ⌘ An upper bound Γ is an <u>*mub*</u> of $\Delta 1$ and $\Delta 2$ in $\langle \mathcal{DT}, \models^{\#/\flat} \rangle$ if $\Gamma \models^{\#/\flat} \Gamma'$ implies $\Gamma' \models^{\#/\flat} \Gamma$ for any upper bound Γ' of $\Delta 1$ and $\Delta 2$.
- ⌘ Γ is a <u>*lower bound*</u> of $\Delta 1$ and $\Delta 2$ in $\langle \mathcal{DT}, \models^{\#/\flat} \rangle$ if $\Delta 1 \models^{\#/\flat} \Gamma$ and $\Delta 2 \models^{\#/\flat} \Gamma$.
- ⌘ A lower bound Γ is an <u>*mlb*</u> of $\Delta 1$ and $\Delta 2$ in $\langle \mathcal{DT}, \models^{\#/\flat} \rangle$ if $\Gamma' \models^{\#/\flat} \Gamma$ implies $\Gamma \models^{\#/\flat} \Gamma'$ for any lower bound Γ' of $\Delta 1$ and $\Delta 2$.

Minimal Upper / Maximal Lower Bounds

- **Theorem**:
  - Γ is an mub of $\Delta 1$ and $\Delta 2$ in $\langle \mathcal{DT}, \models^{\#} \rangle$ iff $E(\Gamma) = min\{ cl(S \cup T) \mid S \in E(\Delta 1), T \in E(\Delta 2) \}$.
  - Γ is an mlb of $\Delta 1$ and $\Delta 2$ in $\langle \mathcal{DT}, \models^{\#} \rangle$ iff $E(\Gamma) = min(E(\Delta 1) \cup E(\Delta 2))$.
  - Γ is an mub of $\Delta 1$ and $\Delta 2$ in $\langle \mathcal{DT}, \models^{\flat} \rangle$ iff $E(\Gamma) = max(E(\Delta 1) \cup E(\Delta 2))$.
  - Γ is an mlb of $\Delta 1$ and $\Delta 2$ in $\langle \mathcal{DT}, \models^{\flat} \rangle$ iff $E(\Gamma) = max\{ S \cap T \mid S \in E(\Delta 1), T \in E(\Delta 2) \}$.
- A top / bottom element of $\langle \mathcal{DT}, \models^{\#} \rangle$ is $\{ \frac{:\neg p}{p} \}$ / $\{\}$.
- A top / bottom element of $\langle \mathcal{DT}, \models^{\flat} \rangle$ is $\{ p, \neg p \}$ / $\{ \frac{:\neg p}{p} \}$.

Skeptical/Credulous Entailment in More/Less General Default Theories

⌘ **Theorem**: T_1 , T_2: first-order theories

- $T_1 \models T_2$ iff $T_1 \models^{\#} T_2$ iff $T_1 \models^{\flat} T_2$.

⌘ **Theorem**: Δ_1 , Δ_2: default theories

- If $\Delta_1 \models^{\#} \Delta_2$ then $skp(\Delta_2) \subseteq skp(\Delta_1)$.
- $\Delta_1 \models^{\flat} \Delta_2$ **iff** $crd(\Delta_2) \subseteq crd(\Delta_1)$.

⌘ Pre-orders $\models_{skp/crd}$ based on skeptical/credulous entailment relations can also be defined. Then, an mub/mlb of Δ_1 and Δ_2 in $\langle \mathcal{DT}, \models^{\#/\flat} \rangle$ is an mub/mlb of Δ_1 and Δ_2 in $\langle \mathcal{DT}, \models_{skp/crd} \rangle$.

Strong Generality

⌘ $\Delta_1, \Delta_2 \in \mathcal{DT}$

- Δ_1 is **strongly more #-general** than Δ_2 :
  $\Delta_1 \trianglerighteq^{\#} \Delta_2$ iff $\Delta_1 \cup \Pi \models^{\#} \Delta_2 \cup \Pi$ for any $\Pi \in \mathcal{DT}$.
- Δ_1 is **strongly more ♭-general** than Δ_2 :
  $\Delta_1 \trianglerighteq^{\flat} \Delta_2$ iff $\Delta_1 \cup \Pi \models^{\flat} \Delta_2 \cup \Pi$ for any $\Pi \in \mathcal{DT}$.

⌘ $\Delta_1 \trianglerighteq^{\#/\flat} \Delta_2$ implies $\Delta_1 \models^{\#/\flat} \Delta_2$.

⌘ $\langle \mathcal{DT}, \trianglerighteq^{\#/\flat} \rangle$ is a pre-ordered set.

⌘**Theorem**: $\Delta_1 \trianglerighteq^{\#} \Delta_2$ and $\Delta_2 \trianglerighteq^{\#} \Delta_1$

iff $\Delta_1 \trianglerighteq^{\flat} \Delta_2$ and $\Delta_2 \trianglerighteq^{\flat} \Delta_1$

iff Δ_1 and Δ_2 are strongly equivalent.

Strong Generality: Example

$$\Delta 1: \quad \frac{:\neg q}{p}$$

$$\Delta 2: \quad \frac{:\neg q}{p}, \quad \frac{:\neg p}{q}$$

$$\Delta 3: \quad p \mid q$$

$$\Delta 4: \quad \frac{:p}{p}, \quad \frac{p:}{q}$$

$E(\Delta 1)=\{cl(\{p\})\}$, $E(\Delta 2)=E(\Delta 3)=\{cl(\{p\}), cl(\{q\})\}$,

$E(\Delta 4)=\{cl(\{p,q\})\}$ $\trianglerighteq^{\#}$ $\trianglerighteq^{\#}$

- $\Delta 1 \trianglerighteq^{\flat} \Delta 2 \trianglerighteq^{\flat} \Delta 3$
- $\Delta 3 \trianglerighteq^{\#/\flat} \Delta 2 \quad \Delta 1$
- No relation holds between Δ4 and others.

Inclusion [Eiter, Tompits & Woltran, IJCAI-05] in Strongly More/Less General Theories

⌘ **Theorem**:

- If $\Delta_1 \trianglerighteq^{\#} \Delta_2$ then $E(\Delta_1) \subseteq E(\Delta_2)$.
- If $\Delta_1 \trianglerighteq^{\flat} \Delta_2$ then $E(\Delta_2) \subseteq E(\Delta_1)$.

⌘ The converse of each does not hold.

⌘ **Theorem (**not in the paper, due to **JianMin Ji)**:

- $\Delta_1 \trianglerighteq^{\#} \Delta_2$ **iff** $E(\Delta_1 \cup \Pi) \subseteq E(\Delta_2 \cup \Pi)$ for any $\Pi \in \mathcal{DT}$. $\trianglerighteq^{\flat}$
- Δ_1 Δ_2 **iff** $E(\Delta_2 \cup \Pi) \subseteq E(\Delta_1 \cup \Pi)$ for any $\Pi \in$

Generality in the Literature

- Generality is often discussed in ILP, but for the FO case only.
- Sakama [IJCAI-2003; TCS 2005]
  - defines an ordering over extended default theories based on multi-valued logics;
  - distinguishes definite and skeptical/credulous default information derived from a program.
  - Equivalent programs do not belong to the same equivalence class induced by Sakama's pre-order. (ex. $\{p\} \geq \{\frac{:p}{p}\}$)
- Eiter, Tompits & Woltran [IJCAI-2005]
  - propose a general framework for comparing logic programs;
  - do not consider generality relations.
- Inoue & Sakama [ICLP 2006]
  - define generality relations for logic programs;
  - do not capture the generality relation in first-order logic.
  - Those relations for logic programs can be viewed as a special case of generality orderings for default logic.

Contributions

- ⌘ A framework to compare generality between disjunctive default theories is proposed and several orderings are defined: **#**- and ♭- generalities and their strong versions.
- ⌘ Both minimal upper and maximal lower bonds can be defined for any pair of default theories in these generality orderings.
- ⌘ **#**-general theories entails more skeptical consequences, while ♭- general theories entails more credulous consequences.
- ⌘ Both (strong) **#**- and (strong) ♭- generalities are defined in a way that (strongly) equivalent theories belong to the same equivalence class induced by these orderings.
- ⌘ These orderings are generalizations of generality relations over first-order theories and those for answer set programming.
- ⌘ The proposed orderings can also be applied to any default semantics in which extensions are guaranteed to be *minimal*.

Future Work

- ⌘ Computing a more (or less) (strongly) general default theory for a given default theory
- ⌘ Computational complexity (Π_P^3-hard?)
- ⌘ Developing generalization/specialization methods in nonmonotonic ILP
- ⌘ Investigating the notion of relative/relativized generality
- ⌘ Extending generality orderings to the class of nested default theories which have non-minimal extensions