Ordering Argumentation Frameworks

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Abstract. This paper introduces two orderings over abstract argumentation frameworks to compare justification status under argumentation semantics. Given two argumentation frameworks AF_1 and AF_2 and an argumentation semantics σ, AF_2 is *more $\sharp$-general than* (or *equal to*) AF_1 (written $AF_1 \sqsubseteq^{\sharp}_{\sigma} AF_2$) if for any σ-extension F of AF_2 there is a σ-extension E of AF_1 such that $E \subseteq F$. In contrast, AF_2 is *more $\flat$-general than* (or *equal to*) AF_1 (written $AF_1 \sqsubseteq^{\flat}_{\sigma} AF_2$) if for any σ-extension E of AF_1 there is a σ-extension F of AF_2 such that $E \subseteq F$. We show that if $AF_1 \sqsubseteq^{\sharp}_{\sigma} AF_2$ then AF_2 skeptically accepts arguments more than AF_1 (under the σ-semantics) while if $AF_1 \sqsubseteq^{\flat}_{\sigma} AF_2$ then AF_2 credulously accepts arguments more than AF_1. Mathematically, these orders constitute pre-order sets over the set of all argumentation frameworks. Next we consider comparing two AFs under dynamic environments by observing the effect of incorporating new information into given AFs. We introduce two orderings in such dynamic environments and show its connection to strong equivalence between argumentation frameworks.

Keywords: argumentation · ordering · strong equivalence

1 Introduction

There are several ways for comparing different theories. Given two first-order theories T_1 and T_2, if $T_1 \models T_2$ holds then every formula derived from T_2 is derived from T_1. In this case, T_1 is considered *more general* (or *informative*) than T_2. For instance, $p \models p \vee q$ means that p is more informative than $p \vee q$. In particular, T_1 is *equivalent* to T_2 ($T_1 \equiv T_2$) if $T_1 \models T_2$ and $T_2 \models T_1$. Inoue and Sakama [7, 8] argue that, in contrast to classical monotonic logic, there is difficulty in defining information ordering in *nonmonotonic logics*. A nonmonotonic theory generally has multiple extensions, and there are two kinds of consequences of a theory, i.e., *skeptical* and *credulous* consequences. This is contrasted to a first-order theory that has a unique extension as the logical consequences of the theory. Then, depending on types of consequences, there exist several definitions for determining that a theory is more informative than another theory. For instance, consider two (nonmonotonic) logic programs: $P_1 = \{\, p \leftarrow not\, q \,\}$ and $P_2 = \{\, p \leftarrow not\, q, \quad q \leftarrow not\, p \,\}$. Then P_1 has the single answer set (or stable model) $\{p\}$ and P_2 has two answer sets $\{p\}$ and $\{q\}$. If we compare skeptical consequences, we can say that P_1 is more informative than P_2 because p is entailed from the former only. Instead, if we compare credulous consequences, P_2 is more informative than P_1 because q is derived from the latter only. As such, the result depends on the type of inference, and in this circumstance information ordering in classical logic cannot be applied. The study [7] then introduces two orderings to logic programs. Given two logic programs P_1 and P_2, $P_1 \models^\sharp P_2$ (P_1 is *more* $\sharp$*-general than* P_2) iff for any answer set S of P_1 there is an answer set T of P_2 such that $T \subseteq S$. Likewise, $P_1 \models^\flat P_2$ (P_1 is *more* $\flat$*-general than* P_2) iff for any answer set T of P_2 there is an answer set S of P_1 such that $T \subseteq S$. These two orderings are respectively called the *Smyth order* and the *Hoare order* in the *domain theory* [6]. The study [7] shows that if $P_1 \models^\sharp P_2$ (resp. $P_1 \models^\flat P_2$) then P_1 entails more skeptical (resp. credulous) consequences than P_2 under the answer set semantics [5]. These orderings are also applied to *default theories* [8] and *abductive theories* [9].

In this paper, we are interested in comparing justification status in *(abstract) argumentation frameworks* (AFs) [3]. Given an argumentation framework AF, an argument x is *skeptically accepted* (or *justified*) under the σ semantics if it is included in every σ-extension of AF, while x is *credulously accepted* if it is included in some σ-extension of AF. The notion of skeptical/credulous justification is of interest in the field of argumentation because "skepticism is related with making more or less committed evaluations about the justification state of arguments in a given situation: more skeptical attitude corresponds to less committed (i.e. more cautious) evaluations" [1]. Baroni and Giacomin [1] then provide systematic comparison of argumentation semantics with respect to their skepticism. They compare skeptical/credulous consequences of different argumentation semantics on a single argumentation framework. In contrast, the current study aims at comparing skeptical/credulous consequences of different argumentation frameworks under the same semantics. Suppose agents (or groups) who have their own argumentation frameworks in which each AF represents an agent's private view of arguments and attack relations. Then it is meaningful to compare those AFs to see which party is more skeptical/credulous in reasoning about arguments. We apply two orderings of [7, 8] to argumentation frameworks and show that those orderings are useful for comparing skeptical/credulous acceptance among different argumentation theories. We also compare AFs under dynamic environments and provide a connection to *strong equivalence* of AFs. The rest of this paper is organized as follows. Section 2 reviews notions used in this paper. Section 3 introduces two orderings between AFs. Section 4 introduces orderings in dynamic environments, and Section 5 addresses final remarks.

2 Preliminaries

2.1 Argumentation Framework

Let $\mathcal{U}$ be the universe of all *arguments*. An *argumentation framework* (AF) [3] is a pair (A, R) where $A \subseteq \mathcal{U}$ is a finite set of arguments and $R \subseteq A \times A$ is the attack relation. The collection of all AFs (induced by $\mathcal{U}$) is denoted by $\mathcal{AF}$. We write $a \rightarrow b$ (a *attacks* b) iff $(a, b) \in R$. A set S of arguments *attacks* an argument a (written $S \rightarrow a$) iff there is an argument $b \in S$ that attacks a. A set S of arguments is *conflict-free* if there are no arguments $a, b \in S$ such that a attacks b. A set S of arguments *defends* an argument a if S attacks every argument that attacks a. We write $D(S) = \{\, a \mid S \text{ defends } a \,\}$. Given $AF = (A, R)$, a conflict-free set of arguments $S \subseteq A$ is:

- an *admissible set* iff $S \subseteq D(S)$;
- a *complete extension* iff $S = D(S)$;
- a *stable extension* iff S attacks each argument in $A \setminus S$;
- a *preferred extension* iff S is a maximal complete extension of AF (wrt $\subseteq$);
- a *grounded extension* iff S is the minimal complete extension of AF (wrt $\subseteq$).

Let $\mathcal{E}^{adm}_{AF}$, $\mathcal{E}^{com}_{AF}$, $\mathcal{E}^{stb}_{AF}$, $\mathcal{E}^{prf}_{AF}$, and $\mathcal{E}^{grd}_{AF}$ be the sets of admissible sets, complete extensions, stable extensions, preferred extensions, and the grounded extension of an AF, respectively. Then the following relations hold:

$$\mathcal{E}^{stb}_{AF} \subseteq \mathcal{E}^{prf}_{AF} \subseteq \mathcal{E}^{com}_{AF} \subseteq \mathcal{E}^{adm}_{AF} \qquad \text{and} \qquad \mathcal{E}^{grd}_{AF} \subseteq \mathcal{E}^{com}_{AF}\,.$$

$\mathcal{E}^{stb}_{AF}$ is possibly empty, while others are not. In particular, $\mathcal{E}^{grd}_{AF}$ is a singleton set. We often write $\mathcal{E}^{\sigma}_{AF}$ where σ means either adm, com, prf, stb or grd. We say that two argumentation frameworks AF_1 and AF_2 are *σ-equivalent* (written $AF_1 \equiv_\sigma AF_2$) if $\mathcal{E}^{\sigma}_{AF_1} = \mathcal{E}^{\sigma}_{AF_2}$. An argument $a \in A$ is *credulously* (resp. *skeptically*) *accepted* under the σ semantics of $AF = (A, R)$ iff $a \in E$ for some (resp. every) $E \in \mathcal{E}^{\sigma}_{AF}$. The set of all credulously (resp. skeptically) accepted arguments under the σ semantics of AF is denoted by $crd^{\sigma}(AF)$ (resp. $skp^{\sigma}(AF)$). When $\mathcal{E}^{stb}_{AF} = \emptyset$, we define $crd^{stb}(AF) = \emptyset$ and $skp^{stb}(AF) = \mathcal{U}$.

2.2 Ordering on Powersets

We recall some mathematical definitions about domains [6]. A *pre-order* (or *quasi-order*) $\preccurlyeq$ is a binary relation which is reflexive and transitive. A pre-order $\preccurlyeq$ is a *partial order* if it is also anti-symmetric. A *pre-ordered set* (resp. *partially ordered set*; *poset*) is a set D with a pre-order (resp. partial order) $\preccurlyeq$ on D. For a pre-ordered set $\langle D, \preccurlyeq \rangle$ and $x, y \in D$, we write $x \prec y$ if $x \preccurlyeq y$ and $y \not\preccurlyeq x$. For a poset $\langle D, \preccurlyeq \rangle$, two elements $x, y \in D$ are *comparable* if $x \preccurlyeq y$ or $y \preccurlyeq x$; otherwise, they are *incomparable*. A *chain* in $\langle D, \preccurlyeq \rangle$ is a subset C of D in which each pair of elements is comparable. An *antichain* in $\langle D, \preccurlyeq \rangle$ is a subset A of D in which each pair of different elements is incomparable, i.e., there is no order relation between any two different elements in A. For a pre-ordered set $\langle D, \preccurlyeq \rangle$ and any set $X \subseteq D$, we denote the maximal and minimal elements of X as follows.

$$min_{\preccurlyeq}(X) = \{\, x \in X \mid \neg\exists y \in X \; s.t. \; y \prec x \,\},$$
$$max_{\preccurlyeq}(X) = \{\, x \in X \mid \neg\exists y \in X \; s.t. \; x \prec y \,\}.$$

We often denote these as $min(X)$ and $max(X)$ by omitting $\preccurlyeq$. We also assume that the relation $\preccurlyeq$ is well-founded (resp. upwards well-founded) on D[3] whenever $min_{\preccurlyeq}(X)$ (resp. $max_{\preccurlyeq}(X)$) is concerned in order to guarantee the existence of a minimal (resp. maximal) element of any $X \subseteq D$. Note that, when D is finite, any pre-order is both well-founded and upwards well-founded on D.

[3] A relation R is *well-founded* on a class D iff every non-empty subset of D has a minimal element with respect to R. A relation R is *upwards well-founded* on D iff the inverse relation R^{-1} is well-founded on D.

For any set D, let $\mathcal{P}(D)$ be the powerset of D. Given a poset $\langle\, D, \preccurlyeq \,\rangle$ and $X, Y \in \mathcal{P}(D)$, two orders are defined as follows:

$$X \preceq^\sharp Y \quad \text{iff} \quad \forall y \in Y\, \exists x \in X \ s.t.\ x \preccurlyeq y\,,$$
$$X \preceq^\flat Y \quad \text{iff} \quad \forall x \in X\, \exists y \in Y \ s.t.\ x \preccurlyeq y\,.$$

The relations $\preceq^\sharp$ and $\preceq^\flat$ are respectively called the *Smyth order* and the *Hoare order*, and both $\langle\, \mathcal{P}(D), \preceq^\sharp \,\rangle$ and $\langle\, \mathcal{P}(D), \preceq^\flat \,\rangle$ are pre-ordered sets.

Example 1. Consider the poset $\langle\, \mathcal{P}(\{p,q\}), \subseteq \,\rangle$. It holds that $\{\{p\},\{q\}\} \preceq^\sharp \{\{p\}\} \preceq^\sharp \{\{p,q\}\}$ and $\{\{p\}\} \preceq^\flat \{\{p\},\{q\}\} \preceq^\flat \{\{p,q\}\}$. Since $\{\emptyset,\{p\}\} \preceq^\sharp \{\emptyset,\{q\}\} \preceq^\sharp \{\emptyset,\{p\}\}$ and $\{\{p\},\{p,q\}\} \preceq^\flat \{\{q\},\{p,q\}\} \preceq^\flat \{\{p\},\{p,q\}\}$ hold, both $\preceq^\sharp$ and $\preceq^\flat$ are not partial orders.

For notational convenience, we often denote two orderings as $\preceq^{\sharp/\flat}$ when distinction between them is unimportant.

3 Ordering Argumentation Frameworks

3.1 Ordering AFs

In this section, we consider a pre-ordered set $\langle\, D, \preccurlyeq \,\rangle$ in which the domain D is $\mathcal{P}(\mathcal{U})$, i.e., the class of sets of arguments in $\mathcal{U}$, and the pre-order $\preccurlyeq$ is the inclusion relation $\subseteq$ over $\mathcal{P}(\mathcal{U})$. In this case $\langle\, \mathcal{P}(\mathcal{U}), \subseteq \,\rangle$ becomes a poset. The Smyth and Hoare orderings on $\mathcal{P}(\mathcal{P}(\mathcal{U}))$ are then defined, which enables us to order classes of sets of arguments.

Definition 1 (orderings over sets of arguments). Let $\langle\, \mathcal{P}(\mathcal{U}), \subseteq \,\rangle$ be a poset. For any Σ_1 and Σ_2 in $\mathcal{P}(\mathcal{P}(\mathcal{U}))$,

$$\Sigma_1 \preceq^\sharp \Sigma_2 \quad \text{iff} \quad \forall T \in \Sigma_2\, \exists S \in \Sigma_1 \ s.t.\ S \subseteq T\,,$$
$$\Sigma_1 \preceq^\flat \Sigma_2 \quad \text{iff} \quad \forall S \in \Sigma_1\, \exists T \in \Sigma_2 \ s.t.\ S \subseteq T\,.$$

Definition 2 (ordering AFs). Let AF_1 and AF_2 be two argumentation frameworks.

$$AF_1 \sqsubseteq^\sharp_\sigma AF_2 \quad \text{iff} \quad \mathcal{E}^\sigma_{AF_1} \preceq^\sharp \mathcal{E}^\sigma_{AF_2},$$
$$AF_1 \sqsubseteq^\flat_\sigma AF_2 \quad \text{iff} \quad \mathcal{E}^\sigma_{AF_1} \preceq^\flat \mathcal{E}^\sigma_{AF_2}$$

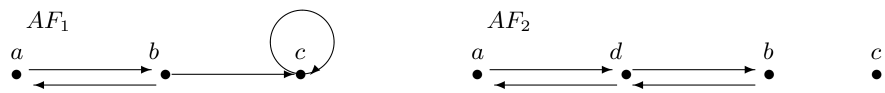
where $\sigma \in \{\, adm, com, prf, stb, grd \,\}$. We say that AF_2 is *more* (or *equally*) $\sharp$-*general* (resp. $\flat$-*general*) than AF_1 (under the σ-semantics) if $AF_1 \sqsubseteq^\sharp_\sigma AF_2$ (resp. $AF_1 \sqsubseteq^\flat_\sigma AF_2$).

We write $AF_1 \equiv^\sharp_\sigma AF_2$ (resp. $AF_1 \equiv^\flat_\sigma AF_2$) iff $AF_1 \sqsubseteq^\sharp_\sigma AF_2$ and $AF_2 \sqsubseteq^\sharp_\sigma AF_1$ (resp. $AF_1 \sqsubseteq^\flat_\sigma AF_2$ and $AF_2 \sqsubseteq^\flat_\sigma AF_1$).

For notational convenience, we often denote two orderings as $\sqsubseteq^{\sharp/\flat}_\sigma$ when distinction between them is unimportant.

Proposition 1. *Let $\mathcal{AF}$ be the collection of all AFs. Then $\langle\, \mathcal{AF}, \sqsubseteq^{\sharp/\flat}_\sigma \,\rangle$ is a pre-ordered set where $\sigma \in \{\, adm, com, prf, stb, grd \,\}$.*

Example 2. Consider $AF_1 = (\{a, b, c\}, \{(a, b), (b, a), (b, c), (c, c)\})$ and $AF_2 = (\{a, b, c, d\}, \{(a, d), (d, a), (b, d), (d, b)\})$.



Then, $\mathcal{E}^{adm}_{AF_1} = \mathcal{E}^{com}_{AF_1} = \{\emptyset, \{a\}, \{b\}\}$, $\mathcal{E}^{prf}_{AF_1} = \{\{a\}, \{b\}\}$, $\mathcal{E}^{stb}_{AF_1} = \{\{b\}\}$, $\mathcal{E}^{grd}_{AF_1} = \{\emptyset\}$; and $\mathcal{E}^{adm}_{AF_2} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{c, d\}, \{a, b, c\}\}$, $\mathcal{E}^{com}_{AF_2} = \{\{c\}, \{c, d\}, \{a, b, c\}\}$, $\mathcal{E}^{prf}_{AF_2} = \mathcal{E}^{stb}_{AF_2} = \{\{c, d\}, \{a, b, c\}\}$, $\mathcal{E}^{grd}_{AF_2} = \{\{c\}\}$. In this case, it holds that $AF_1 \sqsubseteq^{\sharp}_{\sigma} AF_2$ for $\sigma \in \{ adm, com, grd \}$; and $AF_1 \sqsubseteq^{\flat}_{\sigma} AF_2$ for $\sigma \in \{ adm, com, prf, stb, grd \}$.

In what follows, some formal properties are addressed.

Proposition 2. *Let AF_1 and AF_2 be two argumentation frameworks. It holds that (i) $AF_1 \sqsubseteq^{\sharp}_{adm} AF_2$, and (ii) $AF_1 \sqsubseteq^{\sharp}_{grd} AF_2$ iff $AF_1 \sqsubseteq^{\flat}_{grd} AF_2$.*

Proof. For any AF, $\emptyset \in \mathcal{E}^{adm}_{AF}$, and $\mathcal{E}^{grd}_{AF}$ is a singleton set. Hence, the results hold. □

Two relations $\preceq^{\sharp}$ and $\preceq^{\flat}$ are monotonic with respect to the increase of extensions.

Proposition 3. *For any set Σ_1 and Σ_2 in $\mathcal{P}(\mathcal{P}(\mathcal{U}))$, $\Sigma_1 \subseteq \Sigma_2$ implies $\Sigma_1 \preceq^{\flat} \Sigma_2$ and $\Sigma_2 \preceq^{\sharp} \Sigma_1$.*

Proof. If $\Sigma_1 \subseteq \Sigma_2$, then $\forall S \in \Sigma_1$, $S \in \Sigma_2$ thereby $\Sigma_1 \preceq^{\flat} \Sigma_2$ and $\Sigma_2 \preceq^{\sharp} \Sigma_1$. □

Proposition 4. *Let AF_1 and AF_2 be two argumentation frameworks. If $\mathcal{E}^{\sigma}_{AF_1} \subseteq \mathcal{E}^{\sigma}_{AF_2}$ then $AF_1 \sqsubseteq^{\flat}_{\sigma} AF_2$ and $AF_2 \sqsubseteq^{\sharp}_{\sigma} AF_1$ hold for $\sigma \in \{ adm, com, prf, stb, grd \}$.*

Proof. The result follows from Proposition 3. □

Proposition 5. *Let AF_1 and AF_2 be two argumentation frameworks. Then the following results hold for $\sigma \in \{ adm, com, prf, stb, grd \}$.*

(1) $AF_1 \equiv^{\sharp}_{\sigma} AF_2$ *iff* $min_{\subseteq}(\mathcal{E}^{\sigma}_{AF_1}) = min_{\subseteq}(\mathcal{E}^{\sigma}_{AF_2})$.
(2) $AF_1 \equiv^{\flat}_{\sigma} AF_2$ *iff* $max_{\subseteq}(\mathcal{E}^{\sigma}_{AF_1}) = max_{\subseteq}(\mathcal{E}^{\sigma}_{AF_2})$.

Proof. In what follows, $min_{\subseteq}$ is written as min. (1) If $AF_1 \sqsubseteq^{\sharp}_{\sigma} AF_2$, then $\forall S \in min(\mathcal{E}^{\sigma}_{AF_2})\, \exists T \in \mathcal{E}^{\sigma}_{AF_1}$ s.t. $T \subseteq S$, and then $\exists U \in min(\mathcal{E}^{\sigma}_{AF_1})$ s.t. $U \subseteq T$. Thus, $min(\mathcal{E}^{\sigma}_{AF_1}) \preceq^{\sharp} min(\mathcal{E}^{\sigma}_{AF_2})$. Likewise, $AF_2 \sqsubseteq^{\sharp}_{\sigma} AF_1$ implies $min(\mathcal{E}^{\sigma}_{AF_2}) \preceq^{\sharp} min(\mathcal{E}^{\sigma}_{AF_1})$. Assume $min(\mathcal{E}^{\sigma}_{AF_1}) \neq min(\mathcal{E}^{\sigma}_{AF_2})$. Then, (i) $\exists U \in min(\mathcal{E}^{\sigma}_{AF_1}) \setminus min(\mathcal{E}^{\sigma}_{AF_2})$ or (ii) $\exists V \in min(\mathcal{E}^{\sigma}_{AF_2}) \setminus min(\mathcal{E}^{\sigma}_{AF_1})$. In case of (i), $\exists U' \in min(\mathcal{E}^{\sigma}_{AF_2})$ s.t. $U' \subseteq U$ by $min(\mathcal{E}^{\sigma}_{AF_2}) \preceq^{\sharp} min(\mathcal{E}^{\sigma}_{AF_1})$. Also, $\exists U'' \in min(\mathcal{E}^{\sigma}_{AF_1})$ s.t. $U'' \subseteq U'$ by $min(\mathcal{E}^{\sigma}_{AF_1}) \preceq^{\sharp} min(\mathcal{E}^{\sigma}_{AF_2})$. Thus, $U'' \subseteq U$. Since both U and U'' are in $min(\mathcal{E}^{\sigma}_{AF_1})$, $U = U''$ thereby $U' = U$. This contradicts the assumption $U \notin min(\mathcal{E}^{\sigma}_{AF_2})$. Similarly, (ii) also leads to contradiction. Hence, $min(\mathcal{E}^{\sigma}_{AF_1}) = min(\mathcal{E}^{\sigma}_{AF_2})$. (2) is shown in a similar manner. □

Proposition 6. *Let AF_1 and AF_2 be two argumentation frameworks. Then the following three are equivalent for $\sigma \in \{prf, stb, grd\}$: (1) $AF_1 \equiv^{\sharp}_{\sigma} AF_2$, (2) $AF_1 \equiv^{\flat}_{\sigma} AF_2$, (3) $AF_1 \equiv_{\sigma} AF_2$.*

Proof. Consider a poset $\langle \mathcal{P}(\mathcal{U}), \subseteq \rangle$. Since $\mathcal{E}^{\sigma}_{AF}$ is an antichain set for $\sigma \in \{prf, stb, grd\}$, $max_{\subseteq}(\mathcal{E}^{\sigma}_{AF}) = min_{\subseteq}(\mathcal{E}^{\sigma}_{AF}) = \mathcal{E}^{\sigma}_{AF}$. Hence, the result holds by Proposition 5. □

Example 3. Consider $AF_1 = (\{p, q\}, \{(p, q), (q, p), (q, q)\})$ and $AF_2 = (\{p, q\}, \{(p, q), (q, p), (p, p)\})$ where $\mathcal{E}^{com}_{AF_1} = \{\emptyset, \{p\}\}$ and $\mathcal{E}^{com}_{AF_2} = \{\emptyset, \{q\}\}$. Then, $AF_1 \equiv^{\sharp}_{com} AF_2$ but $AF_1 \not\equiv_{com} AF_2$.

Two orderings are related to credulous/skeptical acceptance of arguments.

Proposition 7. *Let AF_1 and AF_2 be two argumentation frameworks. Then the following relations hold for $\sigma \in \{ adm, com, prf, stb, grd \}$.*

1. *If $AF_1 \sqsubseteq^{\flat}_{\sigma} AF_2$ then $crd^{\sigma}(AF_1) \subseteq crd^{\sigma}(AF_2)$.*
2. *If $AF_1 \sqsubseteq^{\sharp}_{\sigma} AF_2$ then $skp^{\sigma}(AF_1) \subseteq skp^{\sigma}(AF_2)$.*

Proof. (1) Assume $AF_1 \sqsubseteq^{\flat}_{\sigma} AF_2$. If $\mathcal{E}^{\sigma}_{AF_1} = \emptyset$ then $crd^{\sigma}(AF_1) = \emptyset$ by definition, and the result holds immediately. Suppose that $\mathcal{E}^{\sigma}_{AF_1} \neq \emptyset$ and $\psi \in crd^{\sigma}(AF_1)$. Then $\psi \in E$ for some $E \in \mathcal{E}^{\sigma}_{AF_1}$. By $AF_1 \sqsubseteq^{\flat}_{\sigma} AF_2$, for any $E \in \mathcal{E}^{\sigma}_{AF_1}$ there is $F \in \mathcal{E}^{\sigma}_{AF_2}$ such that $E \subseteq F$. Then $\psi \in E$ implies $\psi \in F$, thereby $\psi \in crd^{\sigma}(AF_2)$. Hence, $crd^{\sigma}(AF_1) \subseteq crd^{\sigma}(AF_2)$.

(2) Assume $AF_1 \sqsubseteq^{\sharp}_{\sigma} AF_2$. If $\mathcal{E}^{\sigma}_{AF_2} = \emptyset$ then $skp^{\sigma}(AF_2) = \mathcal{U}$ by definition, and the result holds immediately. Suppose that $\mathcal{E}^{\sigma}_{AF_2} \neq \emptyset$. In this case, $\mathcal{E}^{\sigma}_{AF_1} \neq \emptyset$ by $AF_1 \sqsubseteq^{\sharp}_{\sigma} AF_2$. If $\psi \in skp^{\sigma}(AF_1)$ then $\psi \in E$ for every $E \in \mathcal{E}^{\sigma}_{AF_1}$. By $AF_1 \sqsubseteq^{\sharp}_{\sigma} AF_2$, for any $F \in \mathcal{E}^{\sigma}_{AF_2}$ there is $E \in \mathcal{E}^{\sigma}_{AF_1}$ such that $E \subseteq F$. Then $\psi \in E$ implies $\psi \in F$, thereby $\psi \in skp^{\sigma}(AF_2)$. Hence, $skp^{\sigma}(AF_1) \subseteq skp^{\sigma}(AF_2)$. □

Example 4. Consider AFs in Example 2. By $AF_1 \sqsubseteq^{\flat}_{prf} AF_2$, $crd^{prf}(AF_1) = \{a, b\}$ is a subset of $crd^{prf}(AF_2) = \{a, b, c, d\}$. By $AF_1 \sqsubseteq^{\sharp}_{com} AF_2$, $skp^{com}(AF_1) = \emptyset$ is a subset of $skp^{com}(AF_2) = \{c\}$.

By Proposition 7, when $AF_1 \sqsubseteq^{\flat}_{\sigma} AF_2$, AF_2 has more (or equally) credulously accepted arguments than AF_1. In contrast, when $AF_1 \sqsubseteq^{\sharp}_{\sigma} AF_2$, AF_2 has more (or equally) skeptically accepted arguments than AF_1. As such, two orderings over AFs characterize the amount of acceptable arguments in two different modes of reasoning.

3.2 Comparing Different Semantics

In this section, we compare different semantics of a single AF under two orderings. By Proposition 3 and the relations $\mathcal{E}^{stb}_{AF} \subseteq \mathcal{E}^{prf}_{AF} \subseteq \mathcal{E}^{com}_{AF} \subseteq \mathcal{E}^{adm}_{AF}$ and $\mathcal{E}^{grd}_{AF} \subseteq \mathcal{E}^{com}_{AF}$, we have: $\mathcal{E}^{stb}_{AF} \preceq^{\flat} \mathcal{E}^{prf}_{AF} \preceq^{\flat} \mathcal{E}^{com}_{AF} \preceq^{\flat} \mathcal{E}^{adm}_{AF}$, $\mathcal{E}^{grd}_{AF} \preceq^{\flat} \mathcal{E}^{com}_{AF}$, $\mathcal{E}^{adm}_{AF} \preceq^{\sharp} \mathcal{E}^{com}_{AF} \preceq^{\sharp} \mathcal{E}^{prf}_{AF} \preceq^{\sharp} \mathcal{E}^{stb}_{AF}$, and $\mathcal{E}^{com}_{AF} \preceq^{\sharp} \mathcal{E}^{grd}_{AF}$. Moreover, we have the next results.

Proposition 8. *Let AF be an argumentation framework. Then, (1) $\mathcal{E}^{grd}_{AF} \preceq^{\sharp} \mathcal{E}^{\lambda}_{AF}$ for $\lambda \in \{ com, prf, stb, grd \}$, and (2) $\mathcal{E}^{\sigma}_{AF} \preceq^{\flat} \mathcal{E}^{com}_{AF}$ for $\sigma \in \{ adm, com, prf, stb, grd \}$.*

Proof. (1) Since a grounded extension is the least element of $\mathcal{E}^{com}_{AF}$, $\forall E \in \mathcal{E}^{\lambda}_{AF}$, $F \in \mathcal{E}^{grd}_{AF}$ and $F \subseteq E$, thereby $\mathcal{E}^{grd}_{AF} \preceq^{\sharp} \mathcal{E}^{\lambda}_{AF}$. (2) The results $\mathcal{E}^{\sigma}_{AF} \preceq^{\flat} \mathcal{E}^{com}_{AF}$ for $\sigma \in \{ com, prf, stb, grd \}$ is already known. If $E \in \mathcal{E}^{adm}_{AF}$ then $\exists F \in \mathcal{E}^{com}_{AF}$ such that $E \subseteq F$. Hence, $\mathcal{E}^{adm}_{AF} \preceq^{\flat} \mathcal{E}^{com}_{AF}$. □

The above results are combined with the ordering of different AFs. For instance, suppose that $AF_1 \sqsubseteq^{\sharp}_{stb} AF_2$ holds. By $\mathcal{E}^{\sigma}_{AF_1} \preceq^{\sharp} \mathcal{E}^{stb}_{AF_1}$ for $\sigma = \{adm, com, prf, stb, grd\}$, for any stable extension F of AF_1 there is a σ-extension E of AF_1 such that $E \subseteq F$. This means that if AF_2 employs the stable semantics, then AF_2 is more $\sharp$-general than AF_1 that employs any semantics. Suppose, on the other hand, that $AF_1 \sqsubseteq^{\flat}_{\sigma} AF_2$ holds. By $\mathcal{E}^{\sigma}_{AF_2} \preceq^{\flat} \mathcal{E}^{com}_{AF_2}$ (Proposition 8(2)), for any σ-extension E of AF_2 there is a complete extension F of AF_2 such that $E \subseteq F$. This means that if AF_2 employs the complete semantics, then AF_2 is more $\flat$-general than AF_1 that employs any semantics.

3.3 Minimal Upper and Maximal Lower Bounds

In this section, we consider a *minimal upper bound* and a *maximal lower bound* of the sets of extensions with respect to two orderings $\preceq^{\sharp}$ and $\preceq^{\flat}$.

Definition 3 (mub, mlb). Let $\langle \mathcal{P}(\mathcal{P}(\mathcal{U})), \preceq^{\sharp/\flat} \rangle$ be a pre-ordered set. For any Σ_1 and Σ_2 in $\mathcal{P}(\mathcal{P}(\mathcal{U}))$, a set $\Sigma \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$ is an *upper bound* of Σ_1 and Σ_2 if $\Sigma_1 \preceq^{\sharp/\flat} \Sigma$ and $\Sigma_2 \preceq^{\sharp/\flat} \Sigma$. An upper bound Σ is a *minimal upper bound* (*mub*) of Σ_1 and Σ_2 if for any upper bound Σ' of Σ_1 and Σ_2, $\Sigma' \preceq^{\sharp/\flat} \Sigma$ implies $\Sigma \preceq^{\sharp/\flat} \Sigma'$.

On the other hand, a set $\Sigma \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$ is a *lower bound* of Σ_1 and Σ_2 if $\Sigma \preceq^{\sharp/\flat} \Sigma_1$ and $\Sigma \preceq^{\sharp/\flat} \Sigma_2$. A lower bound Σ is a *maximal lower bound* (*mlb*) of Σ_1 and Σ_2 if for any lower bound Σ' of Σ_1 and Σ_2, $\Sigma \preceq^{\sharp/\flat} \Sigma'$ implies $\Sigma' \preceq^{\sharp/\flat} \Sigma$.

Proposition 9. *Let Σ_1 and Σ_2 be two antichain sets in $\langle \mathcal{P}(\mathcal{U}), \subseteq \rangle$.*

1. *$\Sigma \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$ is an mub of Σ_1 and Σ_2 in $\langle \mathcal{P}(\mathcal{P}(\mathcal{U})), \preceq^{\sharp} \rangle$ iff $\Sigma = min_{\subseteq}(X)$ where $X = \{S \cup T \mid S \in \Sigma_1$ and $T \in \Sigma_2\}$.*
2. *$\Sigma \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$ is an mub of Σ_1 and Σ_2 in $\langle \mathcal{P}(\mathcal{P}(\mathcal{U})), \preceq^{\flat} \rangle$ iff $\Sigma = max_{\subseteq}(\Sigma_1 \cup \Sigma_2)$.*
3. *$\Sigma \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$ is an mlb of Σ_1 and Σ_2 in $\langle \mathcal{P}(\mathcal{P}(\mathcal{U})), \preceq^{\sharp} \rangle$ iff $\Sigma = min_{\subseteq}(\Sigma_1 \cup \Sigma_2)$.*
4. *$\Sigma \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$ is an mlb of Σ_1 and Σ_2 in $\langle \mathcal{P}(\mathcal{P}(\mathcal{U})), \preceq^{\flat} \rangle$ iff $\Sigma = max_{\subseteq}(Y)$ where $Y = \{S \cap T \mid S \in \Sigma_1$ and $T \in \Sigma_2\}$.*

Proof. We show (1) and (3). The results of (2) and (4) are shown in similar ways.
(1) Σ is an upper bound of Σ_1 and Σ_2 in $\langle \mathcal{P}(\mathcal{P}(\mathcal{U})), \preceq^{\sharp} \rangle$ iff $\Sigma_1 \preceq^{\sharp} \Sigma$ and $\Sigma_2 \preceq^{\sharp} \Sigma$
iff $\forall S \in \Sigma\ \exists T_1 \in \Sigma_1$ s.t. $T_1 \subseteq S$ and $\forall S \in \Sigma\ \exists T_2 \in \Sigma_2$ s.t. $T_2 \subseteq S$
iff $\forall S \in \Sigma\ \exists T_1 \in \Sigma_1\ \exists T_2 \in \Sigma_2$ s.t. $T_1 \cup T_2 \subseteq S$ $(*)$.

Now suppose that Σ is given as $min_{\subseteq}(\{S \cup T \mid S \in \Sigma_1$ and $T \in \Sigma_2\})$. Σ is an antichain set. Then Σ is an upper bound of Σ_1 and Σ_2 because $(*)$ is satisfied. Assume that Σ is not an mub. Then there is an antichain set[4] $\Gamma \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$ s.t. (i) Γ is an upper bound of Σ_1 and Σ_2, and (ii) $\Gamma \preceq^{\sharp} \Sigma$ and (iii) $\Sigma \not\preceq^{\sharp} \Gamma$. Thus, $\Gamma \neq \Sigma$. For any $U \in \Sigma$, there are $S_1 \in \Sigma_1$ and $T_1 \in \Sigma_2$ s.t. $U = S_1 \cup T_1$ by the definition of Σ. For this U, there is a set $V \in \Gamma$ such that $V \subseteq U$ by (ii) and that $S_2 \cup T_2 \subseteq V$ for some $S_2 \in \Sigma_1$ and $T_2 \in \Sigma_2$ by (i) and $(*)$. So $S_2 \cup T_2 \subseteq S_1 \cup T_1$. Since Σ is the collection of minimal

[4] Without loss of generality, Γ is assumed to be an antichain set. If Γ is not an antichain set, there is $S, T \in \Gamma$ s.t. $S \subseteq T$. Put $\Gamma' = \Gamma \setminus \{T\}$. Then Γ' is an upper bound of Σ_1 and Σ_2 (because if Γ satisfies $(*)$ then Γ' satisfies $(*)$) and also satisfies (ii) and (iii).

sets, $S_2 \cup T_2 = S_1 \cup T_1$. Thus, $U = V$. Hence, $\Sigma \subseteq \Gamma$. By $\Sigma \neq \Gamma$, there is $W \in \Gamma \setminus \Sigma$. Again $S_3 \cup T_3 \subseteq W$ for some $S_3 \in \Sigma_1$ and $T_3 \in \Sigma_2$ by (i) and $(*)$. However, there must be some $X \in \Sigma$ such that $X \subseteq W$ by the construction of Σ and the minimality of Σ. Because $W \notin \Sigma$, $X \subset W$ holds. However, by $\Gamma \preceq^\sharp \Sigma$ there is $Y \in \Gamma$ such that $Y \subseteq X$ and hence $Y \subset W$. This contradicts the fact that Γ is an antichain set.

(3) Σ is a lower bound of Σ_1 and Σ_2 in $\langle \mathcal{P}(\mathcal{P}(\mathcal{U})), \preceq^\sharp \rangle$ iff $\Sigma \preceq^\sharp \Sigma_1$ and $\Sigma \preceq^\sharp \Sigma_2$
iff $\forall S_1 \in \Sigma_1\ \exists T \in \Sigma$ s.t. $T \subseteq S_1$ and $\forall S_2 \in \Sigma_2\ \exists T \in \Sigma$ s.t. $T \subseteq S_2$
iff $\forall S \in \Sigma_1 \cup \Sigma_2\ \exists T \in \Sigma$ s.t. $T \subseteq S$ $(\dagger)$.

Now suppose that $\Sigma = min_{\subseteq}(\Sigma_1 \cup \Sigma_2)$. Then Σ is a lower bound of Σ_1 and Σ_2 because $(\dagger)$ is satisfied. Assume that Σ is not an mlb. Then there is an antichain set $\Gamma \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$ s.t. (i) Γ is a lower bound of Σ_1 and Σ_2, and (ii) $\Sigma \preceq^\sharp \Gamma$ and (iii) $\Gamma \not\preceq^\sharp \Sigma$. Thus, $\Sigma \neq \Gamma$. By (ii), for any $V \in \Gamma$, there is $U \in \Sigma$ such that $U \subseteq V$. By this and the fact that Γ is a lower bound of Σ_1 and Σ_2, we have that $\forall W \in \Sigma_1 \cup \Sigma_2$, $\exists V \in \Gamma\ \exists U \in \Sigma$ such that $U \subseteq V \subseteq W$. As $U \in \Sigma_1 \cup \Sigma_2$, it must be $U = V$ by the minimality of Σ, and thus $\Gamma \subseteq \Sigma$. By $\Sigma \neq \Gamma$, there is $X \in \Sigma \setminus \Gamma$. Since $X \in \Sigma_1 \cup \Sigma_2$ by the construction of Σ, there must be some $Y \in \Gamma$ such that $Y \subseteq X$ by $(\dagger)$. As $X \notin \Gamma$, $Y \subset X$ holds. However, by (ii) there is $Z \in \Sigma$ such that $Z \subseteq Y$ and thus $Z \subset X$. This contradicts the fact that Σ is an antichain set. Therefore, Σ is a mlb of Σ_1 and Σ_2 in $\langle \mathcal{P}(\mathcal{P}(\mathcal{U})), \preceq^\sharp \rangle$. □

Proposition 9 states that an mub or mlb of two antichain sets in $\langle \mathcal{P}(\mathcal{P}(\mathcal{U})), \preceq^{\sharp/\flat} \rangle$ is constructed by the operations min or max. Suppose two argumentation frameworks AF_1 and AF_2 having the sets of σ-extensions $\mathcal{E}^\sigma_{AF_1}$ and $\mathcal{E}^\sigma_{AF_2}$, respectively. Then, a question is whether there is $AF \in \mathcal{AF}$ such that $\mathcal{E}^\sigma_{AF}$ is obtained as an mub (or mlb) of $\mathcal{E}^\sigma_{AF_1}$ and $\mathcal{E}^\sigma_{AF_2}$. If $\sigma = grd$, there is an AF that has the extension obtained as the mub of Proposition 9(1) or (2). This is because if $\mathcal{E}^{grd}_{AF_1} = \{E\}$ and $\mathcal{E}^{grd}_{AF_2} = \{F\}$ then we can construct an AF s.t. $\mathcal{E}^{grd}_{AF} = \{E \cup F\}$ or $\mathcal{E}^{grd}_{AF} = \{E \cap F\}$ as $AF = (E \cup F, \emptyset)$ or $AF = (E \cap F, \emptyset)$. On the other hand, an AF having the grounded extension as the mlb of Proposition 9(3) or (4) does not always exist. This is because $min_{\subseteq}(\mathcal{E}^{grd}_{AF_1} \cup \mathcal{E}^{grd}_{AF_2})$ or $max_{\subseteq}(\mathcal{E}^{grd}_{AF_1} \cup \mathcal{E}^{grd}_{AF_2})$ is not a singleton set in general. When an AF has multiple extensions, the answer is also negative in general.

Example 5. Consider AF_1 and AF_2 such that $\mathcal{E}^{stb}_{AF_1} = \{\{a,b\},\{a,c\}\}$ and $\mathcal{E}^{stb}_{AF_2} = \{\{b,c\}\}$. Then, $min_{\subseteq}(\mathcal{E}^{stb}_{AF_1} \cup \mathcal{E}^{stb}_{AF_2}) = max_{\subseteq}(\mathcal{E}^{stb}_{AF_1} \cup \mathcal{E}^{stb}_{AF_2}) = \{\{a,b\},\{a,c\},\{b,c\}\}$, but there is no AF such that $\mathcal{E}^{stb}_{AF} = \{\{a,b\},\{a,c\},\{b,c\}\}$.

Any stable extension must be incomparable and *tight*, and the set $\{\{a,b\},\{a,c\},\{b,c\}\}$ does not satisfy this condition [2, 4]. As such, the existence of an mub or mlb as a set of extensions as in Proposition 9 does *not* imply that it is *realizable* under a particular semantics [2, 4], that is, it is not necessarily the case that there is an AF having the set of σ-extensions that coincide with an mub or mlb of two sets of extensions of two AFs. Investigating necessary and/or sufficient conditions for the existence of an mub/mlb of two AFs under σ-semantics is left for future study.

4 Strong Ordering

This section considers comparing two AFs under dynamic environments by observing the effect of incorporating new information into given argumentation frameworks. In this section we consider $AF = (A, R)$ where $A \subseteq \mathcal{U}$ and $R \subseteq \mathcal{U} \times \mathcal{U}$.[5] Given $AF_1 = (A_1, R_1)$ and $AF_2 = (A_2, R_2)$, define $AF_1 \sqcup AF_2 = (A_1 \cup A_2, R_1 \cup R_2)$.

Definition 4. Let AF_1 and AF_2 be two argumentation frameworks. Then,

$$AF_1 \trianglelefteq^{\sharp}_{\sigma} AF_2 \;\text{ iff }\; (AF_1 \sqcup AF) \sqsubseteq^{\sharp}_{\sigma} (AF_2 \sqcup AF) \text{ for any } AF \in \mathcal{AF},$$
$$AF_1 \trianglelefteq^{\flat}_{\sigma} AF_2 \;\text{ iff }\; (AF_1 \sqcup AF) \sqsubseteq^{\flat}_{\sigma} (AF_2 \sqcup AF) \text{ for any } AF \in \mathcal{AF}$$

where $\sigma \in \{\, adm, com, prf, stb, grd \,\}$.

We write $\trianglelefteq^{\sharp/\flat}_{\sigma}$ to represent both $\trianglelefteq^{\sharp}_{\sigma}$ and $\trianglelefteq^{\flat}_{\sigma}$ together. The relation $AF_1 \trianglelefteq^{\sharp/\flat}_{\sigma} AF_2$ implies $AF_1 \sqsubseteq^{\sharp/\flat}_{\sigma} AF_2$ by putting $AF = (\emptyset, \emptyset)$.

Proposition 10. *Let AF_1 and AF_2 be two argumentation frameworks. If $AF_1 \trianglelefteq^{\sharp/\flat}_{\sigma} AF_2$ then $AF_1 \sqsubseteq^{\sharp/\flat}_{\sigma} AF_2$ where $\sigma \in \{\, adm, com, prf, stb, grd \,\}$.*

By Proposition 2, the next result holds.

Proposition 11. *Let AF_1 and AF_2 be two argumentation frameworks. Then, (i) $AF_1 \trianglelefteq^{\sharp}_{adm} AF_2$, and (ii) $AF_1 \trianglelefteq^{\sharp}_{grd} AF_2$ iff $AF_1 \trianglelefteq^{\flat}_{grd} AF_2$.*

Two argumentation frameworks AF_1 and AF_2 are *strongly equivalent* (wrt σ semantics) if $AF_1 \sqcup AF \equiv_{\sigma} AF_2 \sqcup AF$ for any $AF \in \mathcal{AF}$ [10]. The notion of strong equivalence is related to the orderings $\trianglelefteq^{\sharp/\flat}_{\sigma}$ as follows.

Proposition 12. *Let AF_1 and AF_2 be two argumentation frameworks. Then the following three are equivalent for $\sigma \in \{\, prf, stb, grd \,\}$: (1) $AF_1 \trianglelefteq^{\sharp}_{\sigma} AF_2 \trianglelefteq^{\sharp}_{\sigma} AF_1$, (2) $AF_1 \trianglelefteq^{\flat}_{\sigma} AF_2 \trianglelefteq^{\flat}_{\sigma} AF_1$, (3) AF_1 and AF_2 are strongly equivalent.*

Proof. $AF_1 \trianglelefteq^{\sharp/\flat}_{\sigma} AF_2 \trianglelefteq^{\sharp/\flat}_{\sigma} AF_1$
iff $(AF_1 \sqcup AF) \sqsubseteq^{\sharp/\flat}_{\sigma} (AF_2 \sqcup AF) \sqsubseteq^{\sharp/\flat}_{\sigma} (AF_1 \sqcup AF)$ for any $AF \in \mathcal{AF}$
iff $(AF_1 \sqcup AF) \equiv_{\sigma} (AF_2 \sqcup AF)$ for any $AF \in \mathcal{AF}$ (Proposition 6)
iff AF_1 and AF_2 are strongly equivalent. □

Example 6. ([10]) Two argumentation frameworks $AF_1 = (\{a, b, c\}, \{(a, b), (b, c), (c, a)\})$ and $AF_2 = (\{a, b, c\}, \{(a, c), (c, b), (b, a)\})$ have the same preferred extension $\emptyset$, but they are not strongly equivalent. This is explained by the fact that for $AF = (\{a, b\}, \{(a, b)\})$, $AF_1 \sqcup AF$ has the preferred extension $\emptyset$, while $AF_2 \sqcup AF$ has the preferred extension $\{a\}$, thereby $(AF_2 \sqcup AF) \not\sqsubseteq^{\sharp}_{prf} (AF_1 \sqcup AF)$.

Proposition 13. *Let AF_1 and AF_2 be two argumentation frameworks. Then the following results hold for $\sigma \in \{\, prf, stb, grd \,\}$.*

[5] We relax the condition by technical reasons but it does not affect the results of previous sections. This is because attack relations in $(\mathcal{U} \times \mathcal{U}) \setminus (A \times A)$ do not change extensions of AF.

1. If $AF_1 \unlhd^\flat_\sigma AF_2$ then $\mathcal{E}^\sigma_{AF_1} \subseteq \mathcal{E}^\sigma_{AF_2}$.
2. If $AF_1 \unlhd^\sharp_\sigma AF_2$ then $\mathcal{E}^\sigma_{AF_2} \subseteq \mathcal{E}^\sigma_{AF_1}$.

Proof. (1) Let $AF_1 = (A_1, R_1)$ and $AF_2 = (A_2, R_2)$. If $AF_1 \unlhd^\flat_\sigma AF_2$, then $AF_1 \sqsubseteq^\flat_\sigma AF_2$ (Proposition 10). Assume $\mathcal{E}^\sigma_{AF_1} \not\subseteq \mathcal{E}^\sigma_{AF_2}$. Then there is an extension $E \in \mathcal{E}^\sigma_{AF_1} \setminus \mathcal{E}^\sigma_{AF_2}$. By $AF_1 \sqsubseteq^\flat_\sigma AF_2$, there is $F \in \mathcal{E}^\sigma_{AF_2}$ such that $E \subset F$. For any F satisfying $E \subset F$, there is an argument $a \in F \setminus E$. Since F is conflict-free, $E \not\rightarrow a$. Suppose that $a \in A_1$. The fact $a \notin E$ implies $a \notin D(E)$. Then there is $(b, a) \in R_1$ s.t. $b \in A_1$ and $E \not\rightarrow b$. Since $(E \not\rightarrow a)$, $b \notin E$ thereby $b \notin D(E)$. Then there is $(c, b) \in R_1$ s.t. $c \in A_1$, $c \neq a$ and $E \not\rightarrow c$. (If $c = a$ then $E' = E \cup \{a\}$ defends every element in E'. So $E' \in \mathcal{E}^\sigma_{AF_1}$ which contradicts the antichain property of $\mathcal{E}^\sigma_{AF_1}$.) Since $(E \not\rightarrow b)$, $c \notin E$ thereby $c \notin D(E)$. Repeating the above argument, A_1 becomes an infinite set. This contradicts the assumption that A_1 is finite. Hence, there is an argument $a \in F \setminus E$ s.t. $a \notin A_1$. Consider $AF = (\{d\}, \{(a, d)\})$ where $d \notin A_1 \cup A_2$. Then $AF_1 \sqcup AF$ has an extension $E' = E \cup \{d\}$, while F is an extension of $AF_2 \sqcup AF$. So $E' \not\subseteq F$. Moreover, for any $G \in \mathcal{E}^\sigma_{AF_2}$ such that $E \not\subset G$, $E' = E \cup \{d\} \not\subseteq G$. Thus, for any extension G' of $AF_2 \sqcup AF$, $E' \not\subseteq G'$. Hence, $(AF_1 \sqcup AF) \not\sqsubseteq^\flat_\sigma (AF_2 \sqcup AF)$, thereby $AF_1 \not\unlhd^\flat_\sigma AF_2$. Contradiction. (2) is shown in a similar manner. □

Proposition 13 shows that $\unlhd^{\sharp/\flat}_\sigma$ provides a sufficient condition for inclusion between the sets of extensions, while $\sqsubseteq^{\sharp/\flat}_\sigma$ provides a necessary condition for it (Proposition 4).

Proposition 14. *Let AF_1 and AF_2 be two argumentation frameworks. Then the following three are equivalent for $\sigma \in \{\, prf, stb, grd \,\}$: (1) $AF_1 \unlhd^\flat_\sigma AF_2$, (2) $AF_2 \unlhd^\sharp_\sigma AF_1$, (3) $\mathcal{E}^\sigma_{AF_1 \sqcup AF} \subseteq \mathcal{E}^\sigma_{AF_2 \sqcup AF}$ for any $AF \in \mathcal{AF}$.*

Proof. We show (1)⇔(3). The relation (2)⇔(3) is shown in a similar way. Suppose $AF_1 \unlhd^\flat_\sigma AF_2$. By definition, $(AF_1 \sqcup AF) \sqsubseteq^\flat_\sigma (AF_2 \sqcup AF)$ for any $AF \in \mathcal{AF}$. Then $(AF_1 \sqcup AF) \sqcup AF' \sqsubseteq^\flat_\sigma (AF_2 \sqcup AF) \sqcup AF'$ for any AF and AF' in $\mathcal{AF}$. So $AF_1 \sqcup AF \unlhd^\flat_\sigma AF_2 \sqcup AF$ for any $AF \in \mathcal{AF}$. By Proposition 13(1), $\mathcal{E}^\sigma_{AF_1 \sqcup AF} \subseteq \mathcal{E}^\sigma_{AF_2 \sqcup AF}$. Conversely, suppose $\mathcal{E}^\sigma_{AF_1 \sqcup AF} \subseteq \mathcal{E}^\sigma_{AF_2 \sqcup AF}$ for any $AF \in \mathcal{AF}$. By Proposition 4, $AF_1 \sqcup AF \sqsubseteq^\flat_\sigma AF_2 \sqcup AF$ for any $AF \in \mathcal{AF}$. Hence, $AF_1 \unlhd^\flat_\sigma AF_2$. □

As such, two relations $\unlhd^\flat_\sigma$ and $\unlhd^\sharp_\sigma$ are symmetric for $\sigma \in \{\, prf, stb, grd \,\}$.

5 Concluding Remarks

We introduced several orderings for comparing sets of extensions in argumentation frameworks. We showed that two orderings $\sqsubseteq^\sharp_\sigma$ and $\sqsubseteq^\flat_\sigma$ are used for comparing skeptical/credulous acceptance of arguments in different argumentation frameworks. Moreover, those relations have connections to inclusion/equivalence relations between sets of extensions. Since argumentation theories are nonmonotonic, some formal properties addressed in this paper have their counterpart in [7–9]. On the other hand, we show that those orderings are used for comparing different semantics of argumentation, which is not considered in the context of default theories or logic programming. The existence of an AF that has a set of extensions as an mub or mlb of given two sets of extensions is not always guaranteed, which is in contrast with the cases of default theories and logic programming where the existence of an mub or mlb is guaranteed. We considered five semantics of AFs in this paper, but the most results obtained in this paper are independent of particular semantics and applied to other semantics as well.

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