Comparing Abductive Theories

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Abstract. This paper introduces two methods for comparing explanation power of different abductive theories. One is comparing *explainability* for observations, and the other is comparing *explanation contents* for observations. Those two measures are represented by generality relations over abductive theories. The generality relations are naturally related to the notion of *abductive equivalence* introduced by Inoue and Sakama. We also analyze the computational complexity of these relations.

1 Introduction

Abduction has been used in many applications of AI including diagnosis, design, updates, and discovery. Abduction is incorporated in problem-solving and programming technologies as *abductive logic programming* [11]. In the process of building knowledge bases, we need to update an abductive theory in accordance with situation change and discovery of surprising facts. For example, to refine an incomplete description, one may need to add more details to a part of the current theory. Such a refinement is expected to ensure that the revised theory is more powerful in abductive reasoning than the previous one. Then, it is important to evaluate abductive theories by comparing abductive power of each theory in such processes.

In predicate logic, comparison of information contents between theories is done by comparing their logical consequences. For example, given two first-order theories T_1 and T_2, T_1 is considered *more informative* than T_2 if $T_2 \models \psi$ implies $T_1 \models \psi$ for any formula ψ, i.e., $T_1 \models T_2$. In this case, it is also said T_1 is *more general* than T_2 [13, 14]. On the other hand, T_1 and T_2 are *equally informative* if $T_1 \models T_2$ and $T_2 \models T_1$, that is, if T_1 and T_2 are *logically equivalent* ($T_1 \equiv T_2$). Recently, Inoue and Sakama considered the generality conditions for *answer set programming* (ASP) [9] and for Reiter's *default logic* [10]. These generality/equivalence relations compare monotonic/nonmonotonic theories in terms of deduction.

The topic of our interest in this paper is how to *compare abductive theories*. That is, we seek conditions under which an abductive theory has more explanation power than another abductive theory. As far as the authors know, no answer to this question is given in the literature of abduction. To understand the problem, suppose that an abductive theory A_1 is defined to be *stronger* than another abductive theory A_2. This might imply that there is a formula which can be explained in the former but cannot be in the latter. Then, we would expect that A_1 has more background knowledge than A_2 or A_1 has more hypotheses than A_2. However, the situation is not so simple because addition of background knowledge may violate the consistency of some combination of hypotheses. Hence, relationships between amounts of background theories and hypotheses need to be analyzed in depth to compare abductive theories precisely.

In this paper, we consider two logical frameworks for abduction, *first-order abduction* and *abductive logic programming* (ALP). Then, we introduce two methods for comparing explanation power of different abductive theories, which were originally introduced by Inoue and Sakama [8] to identify equivalence of two abductive theories. The first one is aimed at comparing *explainability* for observations in different theories, while the second one is aimed at comparing *explanation contents* for observations. Those two comparison measures are represented by generality relations over abductive theories. Moreover, the generality relations can naturally be related to the notion of *abductive equivalence* in [8]. Note that the proposed techniques for first-order abduction can also be applied to comparing frameworks for *explanatory induction* in inductive logic programming.

The rest of this paper is organized as follows. Section 2 introduces two generality relations for comparing abductive first-order theories. Section 3 applies the similar techniques to ALP. Section 4 relates the abductive generality relations to abductive equivalence. Section 5 discusses the complexity issues. Section 6 gives concluding remarks.

2 Generality Relations in First-order Abduction

In this section, we consider abductive theories represented in first-order logic, which have often been used in abduction in AI, e.g., [17]. In this setting, abductive theories are compared by two measures.

Definition 1 Suppose that B and H are sets of first-order formulas, where B represents *background knowledge* and H is a set of *(candidate) hypotheses*. We call a pair (B, H) a (first-order) *abductive theory*. Given a formula O as an *observation*, a set E of formulas belonging to H[3] is an *explanation* of O in (B, H) if $B \cup E \models O$ and $B \cup E$ is consistent. We say that O is *explainable* in (B, H) if it has an explanation in (B, H).

2.1 Comparing Explainability

We first consider a measure for comparing *explainability* between abductive theories.

Definition 2 An abductive theory $A_1 = (B_1, H_1)$ is *more (or equally) explainable than* an abductive theory $A_2 = (B_2, H_2)$, written as $A_1 \geq A_2$, if every observation explainable in A_2 is also explainable in A_1.

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[3] In this paper we do not specify how H is constructed. For example, when hypotheses contain variables, we could just assume that the set H is closed under instantiation. In another case, we could specify the language of H with a bias and then define that any formula which is constructed from H and satisfies the bias *belongs to* H. This latter treatment enables us to deal with comparing theories for *inductive logic programming* (ILP) [14] within the same logical framework as abduction. In any case, we simply denote as $E \subseteq H$ when E is a set of formulas belonging to H.

Example 1 Consider three abductive theories $A_1 = (B_1, H_1)$, $A_2 = (B_2, H_2)$ and $A_3 = (B_3, H_3)$, where

$$\begin{aligned} B_1 &= \{ \mathit{sprinkler\_was\_on} \supset \mathit{grass\_is\_wet} \}, \\ H_1 &= \{ \mathit{sprinkler\_was\_on},\ \mathit{rained\_last\_night} \}, \\ B_2 &= B_1 \cup \{ \mathit{rained\_last\_night} \supset \mathit{grass\_is\_wet} \}, \\ H_2 &= H_1 \cup \{ \neg(\mathit{sprinkler\_was\_on} \supset \mathit{grass\_is\_wet}) \}, \\ B_3 &= B_2 \cup \{ \mathit{grass\_is\_wet} \supset \mathit{shoes\_are\_wet} \}, \\ H_3 &= H_1 \cup \{ \neg(\mathit{sprinkler\_was\_on} \supset \mathit{shoes\_are\_wet}) \}. \end{aligned}$$

Then, $A_3 \geq A_2 \geq A_1$ holds. In fact, every observation explainable in A_i is explainable in A_{i+1} for $i = 1, 2$. Notice that $A_1 \geq A_2$ also holds because $\mathit{rained\_last\_night}$ can be explained by itself in both A_1 and A_2. By contrast, $\mathit{shoes\_are\_wet}$ is explainable in A_3, but is not in either A_1 or A_2, i.e., $A_2 \not\geq A_3$. Note that each additional hypothesis in $H_j \setminus H_1$ for $j = 2, 3$ has no effect in explaining any formula as it cannot be added to B_j without violating the consistency.

We provide a necessary and sufficient condition for the explainable generality relation. In the following, $Th(\Sigma)$ denotes the set of logical consequences of a set Σ of first-order formulas.

Definition 3 An *extension* of an abductive theory $A = (B, H)$ is $Th(B \cup S)$ where S is a maximal set of formulas belonging to H such that $B \cup S$ is consistent. The set of all extensions of A is denoted as $Ext(A)$.

Lemma 1 ([17]) *Let O be a (possibly infinite) set of formulas. There is an explanation that explains every formula in O in (B, H) iff there is an extension X of (B, H) such that $O \subseteq X$.*

Theorem 2 *Let $A_1 = (B_1, H_1)$ and $A_2 = (B_2, H_2)$ be abductive theories. Then, $A_1 \geq A_2$ holds iff for any extension X_2 of A_2, there is an extension X_1 of A_1 such that $X_2 \subseteq X_1$.*

Proof: ($\Leftarrow$) By Lemma 1, if an observation O is explainable in A_2, there is $X_2 \in Ext(A_2)$ such that $O \in X_2$. For any such X_2, there is $X_1 \in Ext(A_1)$ such that $X_2 \subseteq X_1$. Then, $O \in X_1$ and O is explainable in (B_1, H_1) by Lemma 1. Hence, $A_1 \geq A_2$.

($\Rightarrow$) Assume that there is $X_2 \in Ext(A_2)$ such that $X_2 \not\subseteq X_1$ for any $X_1 \in Ext(A_1)$. Pick a formula ψ^i for each ${X_1}^i \in Ext(A_1)$ such that $\psi^i \in (X_2 \setminus {X_1}^i)\ (\neq \emptyset)$, and let O be the set of ψ^i's from every ${X_1}^i$. Then, $O \subseteq X_2$ but $O \not\subseteq X_1$ for any $X_1 \in Ext(A_1)$. By Lemma 1, $\bigwedge_{F \in O} F$ is explainable in A_2 but is not explainable in A_1. Hence, $A_1 \not\geq A_2$. □

There are several classes of abductive theories in which we can see explainable generality holds under some simple conditions.

Proposition 3 (Assumption-freeness) *Suppose two abductive theories $(B_1, \mathcal{L})$ and $(B_2, \mathcal{L})$, where $\mathcal{L}$ is the set of all literals in the underlying language. Then, $(B_1, \mathcal{L}) \geq (B_2, \mathcal{L})$ iff $B_2 \models B_1$.*

Proof: Any extension of an abductive theory $(B_i, \mathcal{L})$ is logically equivalent to a (complete) model of B_i. By Theorem 2, $(B_1, \mathcal{L}) \geq (B_2, \mathcal{L})$ iff, for any model M of B_2, there is a model N of B_1 such that $M \subseteq N$. Because both M and N are complete, $M \subseteq N$ implies $M = N$. Hence, any model of B_2 is a model of B_1. □

Proposition 4 (Semi-monotonicity) *Suppose that (B, H_1) and (B, H_2) are two abductive theories with the same background knowledge. If $H_1 \supseteq H_2$, then $(B, H_1) \geq (B, H_2)$.*

Proof: For any abductive theory (B, H), we can associate a prerequisite-free normal default theory $\Delta = (D_H, B)$, where $D_H = \{ \frac{:h}{h} \mid h \in H \}$. Then there is a 1-1 correspondence between the extensions of Δ (in the sense of Reiter [18]) and $Ext((B, H))$ [17, Theorem 4.1]. By the semi-monotonicity of normal default theories [18, Theorem 3.2], $H_1 \supseteq H_2$ implies that, for any extension F of $\Delta_2 = (D_{H_2}, B)$, there is an extension E of $\Delta_1 = (D_{H_1}, B)$ such that $F \subseteq E$. By Theorem 2, the result holds. □

For abductive theories $A_1 = (B_1, H)$ and $A_2 = (B_2, H)$ with the same hypotheses, $B_1 \models B_2$ implies neither $A_1 \geq A_2$ nor $A_2 \geq A_1$. This explains the name of *semi*-monotonicity in Proposition 4.

Example 2 Suppose the abductive theories $A = (B, H)$ and $A' = (B', H)$ where $B = \{a \wedge b \supset p\}$, $B' = B \cup \{\neg b\}$, and $H = \{a, b\}$. Then, $A' \not\geq A$ because p has the explanation $\{a, b\}$ in A but is not explainable in A'. On the other hand, $A \not\geq A'$ because $\neg b$ has the explanation $\emptyset$ in A' but is not explainable in A.

2.2 Comparing Explanations

We next provide a second measure for comparing abductive theories. This time we compare *explanation contents*.

Definition 4 An abductive theory $A_1 = (B_1, H_1)$ is *more (or equally) explanatory than* an abductive theory $A_2 = (B_2, H_2)$, written as $A_1 \trianglerighteq A_2$, if, for any observation O, every explanation of O in A_2 is also an explanation of O in A_1.

Example 3 For three abductive theories in Example 1, $A_3 \trianglerighteq A_2 \trianglerighteq A_1$ holds. Although $A_1 \geq A_2$ holds, we see that $A_1 \not\trianglerighteq A_2$ because $\{\mathit{rained\_last\_night}\}$ is an explanation of $\mathit{grass\_is\_wet}$ in A_2 but is not in A_1.

It is easy to see that the relation $\trianglerighteq$ is *stronger* than the relation $\geq$, that is, $A_1 \trianglerighteq A_2$ implies $A_1 \geq A_2$. Now we show the necessary and sufficient condition for explanatory generality.

Theorem 5 *Let $A_1 = (B_1, H_1)$ and $A_2 = (B_2, H_2)$ be abductive theories. Then, $A_1 \trianglerighteq A_2$ holds iff $B_1 \models B_2$ and $\mathcal{H}_1 \supseteq \mathcal{H}_2$ hold, where $\mathcal{H}_i = \{ E \subseteq H_i \mid B_i \cup E$ is consistent$\}$ for $i = 1, 2$.*

Proof: Note that any explanation E of an observation O in (B_i, H_i) satisfies that (1) $B_i \cup E \models O$ and (2) $E \in \mathcal{H}_i$.

($\Leftarrow$) Suppose $A_1 \not\trianglerighteq A_2$. Then there exist a formula O and a set E of formulas such that $B_2 \cup E \models O$ and $E \in \mathcal{H}_2$ while $B_1 \cup E \not\models O$ or $E \notin \mathcal{H}_1$. If $B_1 \cup E \not\models O$ holds, we have $B_1 \not\models E \supset O$ and $B_2 \models E \supset O$, which implies $B_1 \not\models B_2$. If $E \notin \mathcal{H}_1$ holds, by $E \in \mathcal{H}_2$ we have $\mathcal{H}_2 \not\subseteq \mathcal{H}_1$. Hence, the result holds.

($\Rightarrow$) Suppose $A_1 \trianglerighteq A_2$. Then for any formula O and any set E of formulas, $B_2 \cup E \models O$ and $E \in \mathcal{H}_2$ imply $B_1 \cup E \models O$ and $E \in \mathcal{H}_1$. By the fact that $B_2 \cup E \models O$ implies $B_1 \cup E \models O$ for any O, we have $B_2 \cup E \models B_1 \cup E$ for any $E \in \mathcal{H}_2 \cap \mathcal{H}_1$. Then, $B_2 \models B_1$ holds when $E = \emptyset$. By the fact that $E \in \mathcal{H}_2$ implies $E \in \mathcal{H}_1$, we also have $\mathcal{H}_2 \subseteq \mathcal{H}_1$. Hence, the result holds. □

Corollary 6 *Let $A_1 = (B_1, H_1)$ and $A_2 = (B_2, H_2)$ be abductive theories. Then, $A_1 \trianglerighteq A_2$ holds iff $B_1 \models B_2$ and $A_1 \geq A_2$ hold.*

Proof: The set $\mathcal{H}_i$ in Theorem 5 contains every subset E of H_i such that $B_i \cup E$ is consistent. $\mathcal{H}_i$ can be characterized by $Ext(A_i)$ as each consistent theory is a subset of some extension. Then, it can be proved that $\mathcal{H}_1 \supseteq \mathcal{H}_2$ iff for any $X_2 \in Ext(A_2)$, there is $X_1 \in Ext(A_1)$ such that $X_2 \subseteq X_1$. Hence, the result follows from Theorem 2. □

Corollary 7 *If $H_1 \supseteq H_2$, then $(B, H_1) \trianglerighteq (B, H_2)$ holds.*

3 Generality Relations in Abductive Logic Programming

In this section, we turn our attention to generality relations in *abductive logic programming* (ALP) [11]. The most significant difference between abduction in first-order logic and ALP is that ALP allows the nonmonotonic negation-as-failure operator not in a background program. When the background program P is nonmonotonic, the fact that $P \cup E$ is consistent for some set E of hypotheses does not necessarily imply that $P \cup E'$ is consistent for $E' \subset E$. Hence comparing abductive power in ALP should be checked in a more naive manner upon each subset of hypotheses.

Definition 5 An *abductive (logic) program* is a pair $\langle P, \Gamma \rangle$ where

- P is a *(logic) program*, which is a set of rules of the form:

$$L_1; \cdots; L_k; not\, L_{k+1}; \cdots; not\, L_l \\ \leftarrow L_{l+1}, \ldots, L_m,\ not\, L_{m+1}, \ldots, not\, L_n \tag{1}$$

  where each L_i is a literal $(n \geq m \geq l \geq k \geq 0)$, and not represents *negation as failure* (NAF). The symbol ; represents disjunction. The left-hand side of the rule is the *head*, and the right-hand side is the *body*. A program containing variables is a shorthand of its ground instantiation.
- Γ is a set of literals, called *abducibles*. Any instance of an abducible is also an abducible.

Logic programs mentioned above belong to the class of *general extended disjunctive programs* (GEDPs) [6]. If any rule of the form (1) in a program P does not contain not in its head, i.e., $k = l$, P is called an *extended disjunctive program* (EDP) [4]. Moreover, if the head of any rule in an EDP P contains no disjunction, i.e., $k = l \leq 1$, P is called an *extended logic program* (ELP). A semantics of a logic program is given by the *answer set semantics* [4, 6].

We denote the set of all ground literals in the language of a program as Lit. For a program P, the set of answer sets of P is denoted as $AS(P)$. When P is an EDP, $AS(P)$ is an *antichain* in 2^{Lit}, that is, for any two answer sets $S_1, S_2 \in AS(P)$, $S_1 \subseteq S_2$ implies $S_1 = S_2$ [4], but this is not the case for a GEDP. A semantics for ALP is given by extending answer sets of the background program with addition of abducibles. Such an extended answer set is called a *belief set*, which has also been called a *generalized stable model* [11].

Definition 6 Let $A = \langle P, \Gamma \rangle$ be an abductive program, and $E \subseteq \Gamma$. A *belief set* of A (with respect to E) is a consistent answer set of the logic program $P \cup E$. The set of all belief sets of A is denoted as $BS(A)$. A set $S \in BS(A)$ is often denoted as S_E when S is a belief set with respect to E.

Definition 7 Let $A = \langle P, \Gamma \rangle$ be an abductive program, and G a conjunction of ground literals called an *observation*. We will often identify a conjunction G with the set of literals in G. A set $E \subseteq \Gamma$ is an *explanation* of G in A if every ground literal in G is true in a belief set of A with respect to E.[4] When G has an explanation in A, G is *explainable* in A.

Note that restrictions in ALP can be removed so that not only literals but rules can be allowed as abducibles and that observations can contain NAF formulas as well as literals. As in the case of first-order abduction, two generality relations are defined for ALP as follows.

[4] This definition provides *credulous explanations*. Alternatively, *skeptical explanations* are defined as $E \subseteq \Gamma$ such that G is true in every belief set of A with respect to E.

Definition 8 Let $A_1 = \langle P_1, \Gamma_1 \rangle$ and $A_2 = \langle P_2, \Gamma_2 \rangle$ be abductive programs, and G an observation. A_1 is *more (or equally) explainable than* A_2, written as $A_1 \geq A_2$, if every observation explainable in A_2 is also explainable in A_1. On the other hand, A_1 is *more (or equally) explanatory than* A_2, written as $A_1 \trianglerighteq A_2$, if, for any observation G, every explanation of G in A_2 is also an explanation of G in A_1.

Example 4 Let $A_1 = \langle P_1, \Gamma \rangle$ and $A_2 = \langle P_2, \Gamma \rangle$ be abductive programs, where $P_1 = \{\, p \leftarrow a,\ a \leftarrow b \,\}$, $P_2 = \{\, p \leftarrow a,\ p \leftarrow b \,\}$, and $\Gamma = \{a, b\}$. Then, $A_1 \geq A_2$ and $A_2 \geq A_1$, while $A_1 \trianglerighteq A_2$ but $A_2 \not\trianglerighteq A_1$. In fact, $\{b\}$ is an explanation of a in A_1, but is not in A_2.

The following results hold for two generality relations.

Theorem 8 *Let $A_1 = \langle P_1, \Gamma_1 \rangle$ and $A_2 = \langle P_2, \Gamma_2 \rangle$ be abductive programs. Then, $A_1 \geq A_2$ holds iff for any belief set S_2 of A_2, there is a belief set S_1 of A_1 such that $S_2 \subseteq S_1$.*

Proof: ($\Leftarrow$) If G is explainable in A_2, there is $S_2 \in BS(A_2)$ such that $G \subseteq S_2$. For any such S_2, there is $S_1 \in BS(A_1)$ such that $S_2 \subseteq S_1$. Then, $G \subseteq S_1$ and G is explainable in A_1. Hence, $A_1 \geq A_2$.

($\Rightarrow$) Assume that there is $S_2 \in BS(A_2)$ such that $S_2 \not\subseteq S_1$ for any $S_1 \in BS(A_1)$. For each ${S_1}^i \in BS(A_1)$, pick a literal L^i such that $L^i \in (S_2 \setminus {S_1}^i)\ (\neq \emptyset)$, and let G be the set of L^i's from every ${S_1}^i$. Then, $G \subseteq S_2$ but $G \not\subseteq S_1$ for any $S_1 \in BS(A_1)$. That is, G is explainable in A_2 but is not in A_1, i.e., $A_1 \not\geq A_2$. □

Theorem 9 *Let $A_1 = \langle P_1, \Gamma_1 \rangle$ and $A_2 = \langle P_2, \Gamma_2 \rangle$ be abductive programs. Then, $A_1 \trianglerighteq A_2$ holds iff for any $E \subseteq \Gamma_2$ and any $S_E \in BS(A_2)$, there is $T_E \in BS(A_1)$ such that $E \subseteq \Gamma_1$ and $S_E \subseteq T_E$.*

Proof: ($\Rightarrow$) Suppose $A_1 \trianglerighteq A_2$. Then, for any observation G and any $E \subseteq \Gamma_2$, the fact that $G \subseteq S_E$ for some $S_E \in BS(A_2)$ implies that $G \subseteq T_E$ for some $T_E \in BS(A_1)$. Thus, $S_E \subseteq T_E$.

($\Leftarrow$) Suppose $S_E \in BS(A_2)$ for any $E \subseteq \Gamma_2$ implies the existence of $T_E \in BS(A_1)$ with $E \subseteq \Gamma_1$ such that $S_E \subseteq T_E$. Then, for any observation G, $G \subseteq S_E$ implies $G \subseteq T_E$. That is, if G has an explanation E in A_2, G has the same explanation E in A_1. □

Theorem 8 and Theorem 9 might look similar, but the condition of the latter is finer-grained than that of the former. In fact, as in the case of first-order abduction, $A_1 \trianglerighteq A_2$ implies $A_1 \geq A_2$.

4 Connection to Abductive Equivalence

In this section, we consider the relationship between the *generality* relations in abduction proposed in this paper and the *equivalence* relations in abduction proposed in the literature. Inoue and Sakama [8] study different types of equivalence relations in abduction: explainable/explanatory equivalence of abductive theories under both first-order abduction and ALP. Pearce *et al.* [16] characterize a part of these problems in the context of equilibrium logic. In the following, an *abductive framework* A means either a first-order abductive theory $A = (B, H)$ or an abductive logic program $A = \langle P, \Gamma \rangle$.

Definition 9 ([8]) Let A_1 and A_2 be abductive frameworks.

1. A_1 and A_2 are *explainably equivalent* if, for any observation O,[5] O is explainable in A_1 iff O is explainable in A_2.
2. A_1 and A_2 are *explanatorily equivalent* if, for any observation O, E is an explanation of O in A_1 iff E is an explanation of O in A_2.

[5] This definition of explainable equivalence for ALP is not exactly the same as that in [8, Definition 4.3]. In [8] an observation is a single ground literal, while we allow a conjunction of ground literals as an observation.

Explainable equivalence requires that two abductive frameworks have the same explainability for any observation. Explainable equivalence may reflect a situation that two programs have different knowledge to derive the same goals. On the other hand, explanatory equivalence assures that two abductive frameworks have the same explanation contents for any observation. Explanatory equivalence is stronger than explainable equivalence: if two abductive frameworks are explanatorily equivalent then they are explainably equivalent.

By Definitions 2, 4, 8, and 9, it is obvious that all generality relations defined in this paper are "anti-symmetric"[6] in the sense that two abductive frameworks are explainably/explanatorily equivalent iff one is both more (or equally) and less (or equally) explainable/explanatory than another at the same time.

Proposition 10 *Let A_1 and A_2 be abductive frameworks.*

1. *A_1 and A_2 are explainably equivalent iff $A_1 \geq A_2$ and $A_2 \geq A_1$.*
2. *A_1 and A_2 are explanatorily equivalent iff $A_1 \trianglerighteq A_2$ and $A_2 \trianglerighteq A_1$.*

With this correspondence and results in previous sections, we can derive either new characterizations of abductive equivalence or new (and simple) proofs of previously presented results. For first-order abduction, the following results can be verified with new proofs.

Proposition 11 *Two first-order abductive theories A_1 and A_2 are explainably equivalent iff $Ext(A_1) = Ext(A_2)$ holds.*

Proposition 12 *For first-order abductive theories $A_1 = (B_1, H_1)$ and $A_2 = (B_2, H_2)$, the following four statements are equivalent.*

1. *A_1 and A_2 are explanatorily equivalent.*
2. *A_1 and A_2 are explainably equivalent and $B_1 \equiv B_2$.*
3. *$B_1 \equiv B_2$ and $\mathcal{H}_1 = \mathcal{H}_2$.*
4. *$B_1 \equiv B_2$ and $H'_1 = H'_2$, where $H'_i = \{ h \in H_i \mid B_i \cup \{h\}$ is consistent $\}$ for $i = 1, 2$.*

For ALP, the next results can be newly obtained. In the following, for any set X, let $max(X) = \{ x \in X \mid \neg\exists y \in X . x \subset y \}$.

Theorem 13 *Let $A_1 = \langle P_1, \Gamma_1 \rangle$ and $A_2 = \langle P_2, \Gamma_2 \rangle$ be abductive programs. Then, A_1 and A_2 are explainably equivalent iff $max(BS(A_1)) = max(BS(A_2))$.*

Proof: (⇒) By Theorem 8, $A_1 \geq A_2$ implies that, for any $S_2 \in max(BS(A_2))$ there exists $S_1 \in BS(A_1)$ such that $S_2 \subseteq S_1$, and then there exists $S'_1 \in max(BS(A_1))$ such that $S_1 \subseteq S'_1$. By $A_2 \geq A_1$, there exists $S'_2 \in BS(A_2)$ such that $S'_1 \subseteq S'_2$, and then there exists $S''_2 \in max(BS(A_2))$ such that $S'_2 \subseteq S''_2$. Then $S_2 \subseteq S''_2$ holds, but because both belong to $max(BS(A_2))$, $S_2 = S''_2$ holds. Hence, $S_2 (= S'_1)$ also belongs to $max(BS(A_1))$, and thus the result holds. (⇐) can be proved by tracing the above proof backward. □

Theorem 14 *Let $A_1 = \langle P_1, \Gamma_1 \rangle$ and $A_2 = \langle P_2, \Gamma_2 \rangle$ be abductive programs. A_1 and A_2 are explanatorily equivalent iff $\mathcal{C}_1 = \mathcal{C}_2$ holds and $max(AS(P_1 \cup E)) = max(AS(P_2 \cup E))$ for any $E \in \mathcal{C}_i$, where $\mathcal{C}_i = \{ E \subseteq \Gamma_i \mid P_i \cup E$ is consistent$\}$ for $i = 1, 2$.*

Proof: (⇒) Suppose that A_1 and A_2 are explanatorily equivalent. By Theorem 9, $A_1 \trianglerighteq A_2$ implies that, for any $E \subseteq \Gamma_2$ and any $S_E \in BS(A_2)$, there is $T_E \in BS(A_1)$ such that $E \subseteq \Gamma_1$ and $S_E \subseteq T_E$. Then, for any $E \subseteq \Gamma_2$ and any $S \in max(AS(P_2 \cup E))$, $E \subseteq \Gamma_1$ and there is $T \in AS(P_1 \cup E)$ such that $S \subseteq T$, and then there is $T' \in max(AS(P_1 \cup E))$ such that $T \subseteq T'$. By $A_2 \trianglerighteq A_1$, there is $S' \in AS(P_2 \cup E)$ such that $T' \subseteq S'$, and then there is $S'' \in max(AS(P_2 \cup E))$ such that $S' \subseteq S''$. Then $S \subseteq S''$ holds and both belong to $max(AS(P_2 \cup E))$, which imply $S = T' = S''$, and thus $S \in max(AS(P_1 \cup E))$. Hence, (1) if $E \subseteq \Gamma_2$ and $P_2 \cup E$ is consistent then $E \subseteq \Gamma_1$ and $P_1 \cup E$ is consistent, and (2) $max(AS(P_2 \cup E)) \subseteq max(AS(P_1 \cup E))$ for any $E \subseteq \Gamma_2$. Similarly, (3) if $E \subseteq \Gamma_1$ and $P_1 \cup E$ is consistent then $E \subseteq \Gamma_2$ and $P_2 \cup E$ is consitent, and (4) $max(AS(P_1 \cup E)) \subseteq max(AS(P_2 \cup E))$ for any $E \subseteq \Gamma_1$. By (1) and (3), $\mathcal{C}_1 = \mathcal{C}_2$ holds. By (2) and (4), $max(AS(P_1 \cup E)) = max(AS(P_2 \cup E))$ holds for any $E \subseteq \Gamma_1$ and for any $E \subseteq \Gamma_2$. Hence, the result follows.

(⇐) can be proved in a similar way. □

Two logic programs P_1 and P_2 are *strongly equivalent with respect to* a rule set $\mathcal{R}$ if $AS(P_1 \cup R) = AS(P_2 \cup R)$ for any logic program $R \subseteq \mathcal{R}$ [7]. This equivalence notion is a restricted version of *strong equivalence* [12], and is called *relative strong equivalence* [7].[7] The next result was originally shown in [8][8] and then was discussed in [16] for EDPs. Now it can be simply proved by the antichain property of $AS(P)$ for any EDP P.

Corollary 15 *Let $A_1 = \langle P_1, \Gamma \rangle$ and $A_2 = \langle P_2, \Gamma \rangle$ be abductive programs with the same hypotheses such that both P_1 and P_2 are EDPs. Also, let $P'_i = P_i \cup \{ \leftarrow L, \neg L \mid L \in Lit\}$ for $i = 1, 2$. Then, A_1 and A_2 are explanatorily equivalent iff P'_1 and P'_2 are strongly equivalent with respect to Γ.*

5 Complexity Results

We show that the computational complexity of deciding generality between abductive theories becomes more complex in general than that of abductive equivalence presented in [8].

Theorem 16 *Let A_1 and A_2 be two propositional abductive theories. Deciding if $A_1 \geq A_2$ is Π^P_3-complete.*

Proof: Let $A_1 = (B_1, H_1)$ and $A_2 = (B_2, H_2)$. We here identify $Ext(A_i)$ with the extensions of the prerequisite-free normal default theory (D_{H_i}, B_i) for $i = 1, 2$ as in the proof of Proposition 4. For any subset $S \subseteq H_2$, checking if $E = Th(B_2 \cup S)$ is an extension of A_2 is coNP-complete [19]. If $E \in Ext(A_2)$ then deciding if there does not exist $F \in Ext(A_1)$ such that $E \subseteq F$ can be determined by checking if the formula $\bigwedge B_2 \wedge \bigwedge S$ belongs to some extension of A_1, which is Σ^P_2-complete [5]. Thus, we can choose $S \subseteq H_2$ in nondeterministic polynomial time with a Σ^P_2-oracle to decide if $A_1 \not\geq A_2$ holds. Hence, the original problem is the complement of this, and belongs to Π^P_3. We omit the proof of Π^P_3-hardness because of the space limitation. □

Theorem 17 *Let A_1 and A_2 be two propositional abductive theories. Deciding if $A_1 \trianglerighteq A_2$ is Π^P_3-complete.*

Proof: Follows from Corollary 6 and Theorem 16. □

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[6] The relations $\geq$ and $\trianglerighteq$ are also preorders, i.e., reflexive and transitive, for both first-order abduction and ALP.

[7] This definition is due to [7], and is slightly different from the notion of *relativized equivalence* in [20, 16]. In [20], P_1 and P_2 are defined as *strongly equivalent relative to* a literal set U iff $AS(P_1 \cup R) = AS(P_2 \cup R)$ for any set R of rules that are constructed using literals in U.

[8] The condition of EDPs was missing in [8, Theorem 4.4]. In fact, only Theorem 14 holds for GEDPs. Moreover, to characterize inconsistent programs in ALP, an EDP having the answer set Lit should be translated to an EDP without an answer set in Corollary 15.

Theorem 18 *Let $A_1 = \langle P_1, \Gamma_1 \rangle$ and $A_2 = \langle P_2, \Gamma_2 \rangle$ be abductive programs. Deciding if $A_1 \geq A_2$ is (i) Π_2^P-complete when P_1 and P_2 are ELPs, and is (ii) Π_3^P-complete when P_1 and P_2 are GEDPs.*

Proof: A computation problem in GEDPs reduces in polynomial time to the corresponding problem in EDPs [6], so we here consider the cases that each P_i is either an ELP or an EDP.

(Membership) For any guess $S \subseteq Lit$, deciding if $S \in BS(A_2)$ is NP-complete for an ELP P_2 (resp. Σ_2^P-complere for an EDP P_2) [2]. For such an S, deciding if there does not exist $T \in BS(A_1)$ such that $S \subseteq T$ can be determined by credulous reasoning that contains S, which is NP-complete for an ELP P_1 (resp. Σ_2^P-complere for an EDP P_1) [2]. Hence, by Theorem 8, $A_1 \not\geq A_2$ can be nondeterministically solvable with two calls to an NP-oracle (resp. a Σ_2^P-oracle). Therefore, the complement is in Π_2^P (resp. Π_3^P).

(Hardness) We prove for the ELP case. Let $\Phi = \forall X \exists Y.\phi$ be a closed QBF, where $\phi = \bigvee_{j=1}^{n} C_j$ is a DNF formula, that is, C_j is a conjunction of literals. Let $A_1 = \langle P_1, \Gamma_1 \rangle$ and $A_2 = \langle P_2, \Gamma_2 \rangle$ be abductive programs such that $P_1 = \{g \leftarrow C_j \mid 1 \leq j \leq n\}$, $\Gamma_1 = X \cup \neg X \cup Y \cup \neg Y$, $P_2 = \{g \leftarrow\}$, and $\Gamma_2 = X \cup \neg X$, where $\neg X = \{\neg x \mid x \in X\}$ and $\neg Y = \{\neg y \mid y \in Y\}$. Note that both P_1 and P_2 are ELPs. We prove that: $A_1 \geq A_2 \Leftrightarrow \Phi$ is valid.

($\Rightarrow$) Suppose $A_1 \geq A_2$. By Theorem 8, for any $S \in BS(A_2)$, there is $T \in BS(A_1)$ such that $S \subseteq T$. In particular, for any $I_X \subseteq X$, there is a belief set $S \in BS(A_2)$ with respect to $I_X \cup \neg(X \setminus I_X)$, and hence $I_X \cup \neg(X \setminus I_X) \subseteq T$ for some $T \in BS(A_1)$. Since $g \in S$, g must be in T too. Then, some C_j ($1 \leq j \leq n$) must be true under $I_X \cup \neg(X \setminus I_X)$ and $I_Y \cup \neg(Y \setminus I_Y)$ for some $I_Y \subseteq Y$. Hence, ϕ is true under such an interpretation. Since I_X was arbitrary, Φ is valid.

($\Leftarrow$) Suppose Φ is valid. Then for any $I_X \subseteq X$, ϕ is true under $I_X \cup \neg(X \setminus I_X)$ and $I_Y \cup \neg(Y \setminus I_Y)$ for some $I_Y \subseteq Y$. Then some C_j is true under this interpretation, and hence g holds. It is easy to see for any $S \in BS(A_2)$ that there is $T \in BS(A_1)$ such that $S \subseteq T$. By Theorem 8, $A_1 \geq A_2$ holds.

For the EDP case, we can apply a transformation of a QBF $\forall X \exists Y \forall Z.\phi$ into a disjunctive program, which is analogous to the one presented in [1, Theorem 3.1] and [2, Lemma 2]. □

Theorem 19 *Let $A_1 = \langle P_1, \Gamma_1 \rangle$ and $A_2 = \langle P_2, \Gamma_2 \rangle$ be abductive programs. Deciding if $A_1 \trianglerighteq A_2$ is (i) Π_2^P-complete when P_1 and P_2 are ELPs, and is (ii) Π_3^P-complete when P_1 and P_2 are GEDPs.*

Proof: Like Theorem 18, we can assume that each P_i is either an ELP or an EDP. For any guess $S \subseteq Lit$, deciding if $S_E \in BS(A_2)$ for some $E \subseteq \Gamma_2$ is NP-complete for an ELP P_2 (resp. Σ_2^P-complere for an EDP P_2) [2]. For any such E, deciding if $AS(P_1 \cup E) \neq \emptyset$ is NP-complete for an ELP P_2 (resp. Σ_2^P-complere for an EDP P_2) [1]. For S_E, deciding if there does not exist $T \in AS(P_1 \cup E)$ such that $S_E \subseteq T$ can be determined by credulous reasoning that contains S_E, which is NP-complete for an ELP P_1 (resp. Σ_2^P-complere for an EDP P_1) [2]. Hence, by Theorem 9, $A_1 \not\trianglerighteq A_2$ can be nondeterministically solvable with three calls to an NP-oracle (resp. a Σ_2^P-oracle). Therefore, the complement is in Π_2^P (resp. Π_3^P). The hardness can be shown in the same way as in Theorem 18. □

6 Discussion

The relation $\geq$ introduced in this paper can be represented by generality relations defined by Inoue and Sakama [9, 10]. We briefly sketch the relationships here. For first-order abductive theories $A_1 = (B_1, H_1)$ and $A_2 = (B_2, H_2)$, by identifying $Ext(A_i)$ with the extensions of the prerequisite-free normal default theory (D_{H_i}, B_i) for $i = 1, 2$, we can prove that $A_1 \geq A_2$ iff $A_1 \models^{\flat}_{dt} A_2$, where $\models^{\flat}_{dt}$ is a Hoare order defined on the class of default theories [10]. On the other hand, for abductive logic programs $A_1 = \langle P_1, \Gamma_1 \rangle$ and $A_2 = \langle P_2, \Gamma_2 \rangle$, let P'_i ($i = 1, 2$) be the GEDP defined by

$$P'_i = P_i \cup \{\, l; not\, l \leftarrow \ \mid l \in \Gamma_i \}.$$

Then, $BS(A_i) = AS(P'_i)$ holds [6]. With this result, we can see that $A_1 \geq A_2$ iff $P'_1 \models^{\flat}_{lp} P'_2$, where $\models^{\flat}_{lp}$ is a Hoare order defined on the class of GEDPs (originally defined on the class of EDPs in [9]).

Besides work on generality relations in ASP [9], a general correspondence framework has been proposed in [3, 15] to compare logic programs. This framework is defined to compare *equivalence* and *inclusion* between the semantics of logic programs instead of generality, but the notions of *projection* and *contexts* are also introduced to enable a variety of equivalence comparison. Incorporating these notions into our generality framework is a topic of future work.

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