Generality and Equivalence Relations in Default Logic

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Abstract

Generality or refinement relations between different theories have important applications to generalization in inductive logic programming, refinement of ontologies, and coordination in multi-agent systems. We study generality relations in disjunctive default logic by comparing the amounts of information brought by default theories. Intuitively, a default theory is considered *more general* than another default theory if the former brings more information than the latter. Using techniques in domain theory, we introduce different types of generality relations over default theories. We show that generality relations based on the Smyth and Hoare orderings reflect orderings on skeptical and credulous consequences, respectively, and that two default theories are equivalent if and only if they are equally general under these orderings. These results naturally extend both generality relations over first-order theories and those for answer set programming.

Introduction

Comparing the amounts of information brought by different theories is recognized important in knowledge representation to assess the relative value of each theory. Such inter-theory relations have recently attracted more interests in design and maintenance of knowledge bases and ontologies. We give three ontology-related examples as such tasks.

- One has to add more details to a part of incomplete description that has not yet been sufficiently described. The result should be a *refinement* of the former description (Antoniou and Kehagias 2000; Ghilardi *et al.* 2006). The opposite concept of refinement is *abstraction*.
- One may wish to define or determine the *equivalence* between different descriptions or their parts. The notion of equivalence is related to verification, simplification, optimization and modularity of descriptions.
- Integration, merging, or coordination of multiple descriptions is important in dynamic or multi-agent contexts, and one should modify her own description in accordance with those by other agents. One possible way to construct such a description is to take a *minimal generalization* or a *maximal specialization* of those multiple descriptions.

It is thus essential to precisely define inter-theory relations such as *generality* (or *refinement*) as well as *equivalence* relations in knowledge representation.

Intuitively, a theory T_1 is considered *more general* than a theory T_2 if T_1 brings more information than T_2. In first-order logic, T_1 can be defined to be *more general than (or equal to)* T_2 if every formula derived from T_2 is derived from T_1, i.e., $T_1 \models T_2$. This notion has been used as a basis of generalization in *inductive logic programming* (Plotkin 1970; Niblett 1988; Nienhuys-Cheng and de Wolf 1997). On the other hand, *equivalence* of two first-order theories T_1 and T_2 is given by the basic relation $T_1 \equiv T_2$, then two equivalent first-order theories are more general than each other.

Nonmonotonic formalisms such as *default logic* (Reiter 1980) and logic programs with the *answer set semantics* (Gelfond and Lifschitz 1991) are useful for representing incomplete knowledge and partial information, and have been applied to ontologies (Baader and Hollunder 1995; Eiter *et al.* 2003) and multi-agent systems. Then, equivalence of nonmonotonic theories has recently become an active research area. The notion of *strong equivalence* has been explored for the answer set semantics by Lifschitz *et al.* (2001), and for default logic by Turner (2001). On the other hand, relatively fewer studies exist for determining generality in nonmonotonic theories although the equivalence notion has been studied in depth. Sakama (2005) shows a method for ordering default theories by providing a multi-valued interpretation for a default theory. Recently, Inoue and Sakama (2006) study generality relations over the powerset of literals for the answer set semantics.

In contrast to classical monotonic logic, there is a difficulty in defining generality relations in nonmonotonic logics. A default theory generally has multiple extensions, and hence there are two kinds of consequences of a default theory, i.e., *skeptical* and *credulous* consequences. Depending on types of consequences, there exist several definitions for determining that a theory is more general than another theory. This is contrasted to a first-order theory that has a unique extension as the logical consequences of the theory. For instance, consider two default theories:

$$\Delta_1: \quad \frac{:p}{p}\,,$$

$$\Delta_2: \quad \frac{:\neg q}{p}\,,\quad \frac{:\neg p}{q}\,.$$

Here, Δ_1 has the single extension $cl(\{p\})$ and Δ_2 has two extensions $cl(\{p\})$ and $cl(\{q\})$. If we compare skeptical consequences, we can say that Δ_1 is more informative than Δ_2 because p is entailed from the former only. Instead, if we compare credulous consequences, Δ_2 is more informative than Δ_1 because q is derived from the latter only.

With this background and motivation, this paper studies a theory of generality orderings in *disjunctive default logic* (Gelfond *et al.* 1991), which is a generalization of Reiter's default logic (Reiter 1980). Because of existence of multiple extensions in each default theory, it is more natural to compare the collections of extensions of default theories instead of comparing the sets of skeptical/credulous consequences. For this purpose, we will define two generality relations over disjunctive default theories based on *domain theory* (Plotkin 1976; Gunter and Scott 1990). The proposed $\sharp$- and $\flat$-generality orderings are defined with the *Smyth* and *Hoare orderings*, respectively. For the previous example, it holds that (1) $\Delta_1 \models^\sharp \Delta_2$ meaning that, for each extension E of Δ_1, there is an extension E' of Δ_2 such that $E' \subseteq E$, and that (2) $\Delta_2 \models^\flat \Delta_1$ meaning that, for each extension E of Δ_1, there is extension E' of Δ_2 such that $E \subseteq E'$.

These two orderings can be used to refine or abstract descriptions of agents' beliefs and knowledge. For example, suppose that there are three descriptions on food preference represented by the following disjunctive theories:

$$\begin{array}{ll} P_A: & italian \mid american, \\ P_B: & pasta \mid pizza, \\ P_C: & pasta \mid sandwich \mid curry. \end{array}$$

Assume also that taxonomic knowledge is given by:

$$\begin{array}{ll} Ont: & pasta \rightarrow italian,\ \ pizza \rightarrow italian, \\ & sandwich \rightarrow american,\ \ curry \rightarrow indian, \\ & italian \rightarrow food,\ \ american \rightarrow food,\ \ indian \rightarrow food. \end{array}$$

Let $T_A = P_A \cup Ont$, $T_B = P_B \cup Ont$, and $T_C = P_C \cup Ont$. Then, $T_B \models^\sharp T_A$ holds, and T_B is more informative than T_A in the sense that both *pasta* and *pizza* in T_B provide details of *italian* in T_A and that T_B rules out the possibility of *american* in T_A. On the other hand, $T_C \models^\flat T_A$ holds, and T_C is more informative than T_A in the sense that each choice in T_A is detailed in T_C and that T_C provides a further enumeration of *indian* food choices that are missing in T_A.

In this paper, we will see that these proposed orderings have the following desirable properties.

- The $\sharp$- and $\flat$-generality orderings reflect orderings on the skeptical and credulous consequences, respectively.
- Under the $\sharp$- and $\flat$-generality orderings, two default theories are *equivalent*, i.e., have the same extensions, if and only if they are equally general. This property does not hold in the framework by (Sakama 2005).
- Both $\sharp$- and $\flat$-generality orderings are generalizations of generality relations over first-order theories (Niblett 1988; Nienhuys-Cheng and de Wolf 1997) and those for the answer set semantics (Inoue and Sakama 2006). The former generalization result does not hold in the logic programming framework by (Inoue and Sakama 2006).
- Under the $\sharp$- and $\flat$-generality orderings, both a minimal upper bound and a maximal lower bound exist for any pair of default theories. These bounds can be used to define the theoretical solutions for integrating, merging and coordinating multiple default theories.

We also introduce *strong generality* between default theories based on the $\sharp$- and $\flat$-generality orderings, and relate them with *strong equivalence* by (Turner 2001). Surprisingly, strong generality turns out to imply *inclusion*, that is, an extension of one theory is an extension of another theory.

Disjunctive Default Theories

We first review the definition of *disjunctive default logic* (Gelfond *et al.* 1991). A *(disjunctive) default theory* Δ is a set of *(disjunctive) defaults* of the form:

$$\frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma_1 \mid \cdots \mid \gamma_n}$$

where α, $\beta_1, \ldots, \beta_m$ and $\gamma_1, \ldots, \gamma_n$ are quantifier-free first-order formulas and are called the *prerequisite*, the *justifications* and the *consequents*, respectively. Reiter's default logic (Reiter 1980) corresponds to the case when there is exactly one consequent ($n = 1$) in each default, which is called *non-disjunctive*. When $\alpha = true$, a default is called *prerequisite-free* and is written by dropping α. A default is *justification-free* if $m = 0$. A prerequisite-free and justification-free default is written as $\gamma_1 \mid \cdots \mid \gamma_n$; if, in addition, $n = 1$, then a default of the form γ is identified with the first-order formula γ. A default with variables is a shorthand for the set of all its ground instances obtained by substituting variables with the ground terms from the language of Δ. Throughout the paper, we assume a default theory which is already ground-instantiated. Also, a formula means a propositional formula unless stated otherwise.

Let Δ be a default theory and E be a set of formulas. Then, let Δ^E be the justification-free default theory:

$$\Delta^E = \{ \frac{\alpha :}{\gamma_1 \mid \cdots \mid \gamma_n} \mid \frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma_1 \mid \cdots \mid \gamma_n} \in \Delta \text{ and } \neg\beta_1, \ldots, \neg\beta_m \notin E \}.$$

The set E' of formulas is *closed under* the rules from Δ^E if, for any default $\frac{\alpha:}{\gamma_1 \mid \cdots \mid \gamma_n} \in \Delta^E$, if $\alpha \in E'$ then at least one of $\gamma_1, \ldots, \gamma_n$ belongs to E'. Then, a set E of formulas is an *extension* of Δ iff E is minimal among sets of formulas closed under provability in propositional logic and under the rules from Δ^E. A default theory Δ is *consistent* if it has a consistent extension; otherwise, Δ is *inconsistent*. An inconsistent default theory Δ is called *contradictory* if it has the single extension $\mathcal{L}$, which is the set of all formulas in the language of Δ, and is called *incoherent* if it has no extension. Given a default theory Δ, the set of all extensions of Δ is denoted by $Ext(\Delta)$. Note that for any two extensions $E, F \in Ext(\Delta)$, $E \subseteq F$ implies $E = F$. Given two default theories, the following two notions of equivalence are known.

Definition 1 (Turner 2001) Let Δ_1 and Δ_2 be two default theories. Then, Δ_1 and Δ_2 are *(weakly) equivalent* if $Ext(\Delta_1) = Ext(\Delta_2)$ holds. On the other hand, Δ_1 and

Δ_2 are *strongly equivalent* if $Ext(\Delta_1 \cup \Pi) = Ext(\Delta_2 \cup \Pi)$ holds for any default theory Π.[1]

Example 1 $\Delta_1 = \{ \frac{:\neg q}{p} \}$ and $\Delta_2 = \{ p \}$ have the same extension $cl(\{p\})$, where cl is the deductive closure operator, hence they are weakly equivalent. Δ_1 and Δ_2 are not strongly equivalent because $\Delta_1 \cup \{ \frac{p:}{q} \}$ has no extension but $\Delta_2 \cup \{ \frac{p:}{q} \}$ has the extension $cl(\{p, q\})$.

Generality Relations over Default Theories

We first recall some mathematical definitions about domains (Gunter and Scott 1990). A *pre-order* $\sqsubseteq$ is a binary relation which is reflexive and transitive. A pre-order $\sqsubseteq$ is a *partial order* if it is also anti-symmetric. A *pre-ordered set* (resp. *partially ordered set*; *poset*) is a set D with a pre-order (resp. partial order) $\sqsubseteq$ on D.

For any set D, let $\mathcal{P}(D)$ be the powerset of D. Given a poset $\langle D, \sqsubseteq \rangle$ and $X, Y \in \mathcal{P}(D)$, the *Smyth order* is defined as

$$X \models^\sharp Y \quad \text{iff} \quad \forall x \in X\, \exists y \in Y.\, y \sqsubseteq x\,,$$

and the *Hoare order* is defined as

$$X \models^\flat Y \quad \text{iff} \quad \forall y \in Y\, \exists x \in X.\, y \sqsubseteq x\,.$$

Both $\langle \mathcal{P}(D), \models^\sharp \rangle$ and $\langle \mathcal{P}(D), \models^\flat \rangle$ are pre-ordered sets. Note that the orderings $\models^\sharp$ and $\models^\flat$ are slightly different from the standard ones: we allow $\emptyset$ ($\in \mathcal{P}(D)$) as the top element $\top^\sharp$ in $\langle \mathcal{P}(D), \models^\sharp \rangle$ and as the bottom element $\bot^\flat$ in $\langle \mathcal{P}(D), \models^\flat \rangle$. This is because we will associate $\emptyset$ with the class of incoherent default theories so that we enable comparison of all classes of default theories.

Example 2 Consider the poset $\langle \mathcal{P}(\{p, q\}), \subseteq \rangle$. Then, we have $\{\{p, q\}\} \models^\sharp \{\{p\}\}$ and $\{\{p\}\} \models^\sharp \{\{p\}, \{q\}\}$, and hence $\{\{p, q\}\} \models^\sharp \{\{p\}, \{q\}\}$. On the other hand, $\{\{p, q\}\} \models^\flat \{\{p\}, \{q\}\}$ but $\{\{p\}, \{q\}\} \models^\flat \{\{p\}\}$. Note that both $\{\emptyset, \{p\}\} \models^\sharp \{\emptyset, \{q\}\}$ and $\{\emptyset, \{q\}\} \models^\sharp \{\emptyset, \{p\}\}$ hold, indicating that $\models^\sharp$ is not a partial order.

Now, we assume a poset $\langle D, \sqsubseteq \rangle$ such that the domain $D = \mathcal{P}(\mathcal{L})$ is the family of subsets of $\mathcal{L}$, i.e., the class of sets of propositional formulas in the language and the partial-order $\sqsubseteq$ is the subset relation $\subseteq$. Then, we define the Smyth and Hoare orderings on $\mathcal{P}(\mathcal{P}(\mathcal{L}))$, which enables us to order sets of formulas or sets of extensions. In particular, both $\langle \mathcal{P}(\mathcal{P}(\mathcal{L})), \models^\sharp \rangle$ and $\langle \mathcal{P}(\mathcal{P}(\mathcal{L})), \models^\flat \rangle$ are pre-ordered sets. Moreover, if we associate a default theory Δ with its set of extensions $Ext(\Delta)$, the ordering on default theories becomes possible as follows.

Definition 2 Given the poset $\langle \mathcal{P}(\mathcal{L}), \subseteq \rangle$ and two default theories Δ_1 and Δ_2, we define:

$$\Delta_1 \models^\sharp \Delta_2 \quad \text{iff} \quad Ext(\Delta_1) \models^\sharp Ext(\Delta_2),$$

$$\Delta_1 \models^\flat \Delta_2 \quad \text{iff} \quad Ext(\Delta_1) \models^\flat Ext(\Delta_2),$$

$$\Delta_1 \models^\natural \Delta_2 \quad \text{iff} \quad \Delta_1 \models^\sharp \Delta_2 \text{ and } \Delta_1 \models^\flat \Delta_2.$$

We say that Δ_1 is *more $\sharp$-general* (resp. *more $\flat$-general*) *than (or equal to)* Δ_2 if $\Delta_1 \models^\sharp \Delta_2$ (resp. $\Delta_1 \models^\flat \Delta_2$).

[1]Turner (2001) introduces the notion in the context of *nested default logic*. Strong equivalence in default logic is a generalization of that in logic programs (Lifschitz *et al.* 2001).

Intuitively, $\sharp$-generality and $\flat$-generality reflects the following situations. $\Delta_1 \models^\sharp \Delta_2$ means that every extension of Δ_1 is more informative than some extension of Δ_2. On the other hand, $\Delta_1 \models^\flat \Delta_2$ means that every extension of Δ_2 is less informative than some extension of Δ_1. When both Δ_1 and Δ_2 have a single extension, it is obvious that $\Delta_1 \models^\sharp \Delta_2$ iff $\Delta_1 \models^\flat \Delta_2$. The ordering $\natural$ is called the *Plotkin order* (Plotkin 1976) (or *Egli-Milner order*).

Both $\sharp$-generality and $\flat$-generality are naturally connected to the notion of weak equivalence in default logic.

Theorem 1 *Let Δ_1 and Δ_2 be two default theories. Then, the following three are equivalent:*

(1) $\Delta_1 \models^\sharp \Delta_2$ *and* $\Delta_2 \models^\sharp \Delta_1$*;*

(2) $\Delta_1 \models^\flat \Delta_2$ *and* $\Delta_2 \models^\flat \Delta_1$*;*

(3) Δ_1 *and* Δ_2 *are weakly equivalent.*

Proof. We prove (1)⇔(3) but (2)⇔(3) can be proved in the same way.

$\Delta_1 \models^\sharp \Delta_2$ and $\Delta_2 \models^\sharp \Delta_1$
iff $\forall E \in Ext(\Delta_1), \exists F \in Ext(\Delta_2)$ s.t. $F \subseteq E$ and $\forall F \in Ext(\Delta_2), \exists E \in Ext(\Delta_1)$ s.t. $E \subseteq F$
iff $\forall E \in Ext(\Delta_1), \exists F \in Ext(\Delta_2)\, \exists E' \in Ext(\Delta_1)$ s.t. $E' \subseteq F \subseteq E$ and $\forall F \in Ext(\Delta_2), \exists E \in Ext(\Delta_1)\, \exists F' \in Ext(\Delta_2)$ s.t. $F' \subseteq E \subseteq F$
iff $\forall E \in Ext(\Delta_1), \exists F \in Ext(\Delta_2)$ s.t. $F = E$ and $\forall F \in Ext(\Delta_2), \exists E \in Ext(\Delta_1)$ s.t. $E = F$
iff $\forall E \in Ext(\Delta_1), E \in Ext(\Delta_2)$ and $\forall F \in Ext(\Delta_2), F \in Ext(\Delta_2)$ iff $Ext(\Delta_1) \subseteq Ext(\Delta_2)$ and $Ext(\Delta_2) \subseteq Ext(\Delta_1)$ iff $Ext(\Delta_1) = Ext(\Delta_2)$. □

Example 3 Consider the following default theories:

$$\Delta_1: \quad \frac{:\neg q}{p},$$

$$\Delta_2: \quad \frac{:\neg p}{q}, \quad \frac{:\neg q}{p},$$

$$\Delta_3: \quad p \mid q,$$

$$\Delta_4: \quad \frac{:p}{p}, \quad \frac{p:}{q},$$

where $Ext(\Delta_1) = \{ cl(\{p\}) \}$, $Ext(\Delta_2) = Ext(\Delta_3) = \{ cl(\{p\}), cl(\{q\}) \}$, and $Ext(\Delta_4) = \{ cl(\{p, q\}) \}$. Then, $\Delta_4 \models^\sharp \Delta_1 \models^\sharp \Delta_2$, and $\Delta_4 \models^\flat \Delta_2 \models^\flat \Delta_1$. Δ_2 and Δ_3 are weakly equivalent, and thus $\Delta_2 \models^\sharp \Delta_3 \models^\sharp \Delta_2$ and $\Delta_2 \models^\flat \Delta_3 \models^\flat \Delta_2$.

Both a minimal upper bound and a maximal lower bound of any pair of default theories exist with respect to generality orderings. Those bounds are important in the theory of generalization and specialization in inductive logic programming (Plotkin 1970). In the following, let $\mathcal{DT}$ be the class of all default theories which can be constructed in the language. We write $\models^{\sharp/\flat}$ to denote either the $\sharp$- or the $\flat$-generality relation. Then, $\langle \mathcal{DT}, \models^{\sharp/\flat} \rangle$ is a pre-ordered set.

Definition 3 A default theory $\Gamma \in \mathcal{DT}$ is an *upper bound* of $\Delta_1, \Delta_2 \in \mathcal{DT}$ in $\langle \mathcal{DT}, \models^{\sharp/\flat} \rangle$ if $\Gamma \models^{\sharp/\flat} \Delta_1$ and $\Gamma \models^{\sharp/\flat} \Delta_2$. An upper bound Γ is a *minimal upper bound* (*mub*) of Δ_1 and Δ_2 in $\langle \mathcal{DT}, \models^{\sharp/\flat} \rangle$ if for any upper bound Γ' of Δ_1 and Δ_2, $\Gamma \models^{\sharp/\flat} \Gamma'$ implies $\Gamma' \models^{\sharp/\flat} \Gamma$.

On the other hand, $\Gamma \in \mathcal{DT}$ is a *lower bound* of Δ_1 and Δ_2 in $\langle \mathcal{DT}, \models^{\sharp/\flat} \rangle$ if $\Delta_1 \models^{\sharp/\flat} \Gamma$ and $\Delta_2 \models^{\sharp/\flat} \Gamma$. A lower bound Γ is a *maximal lower bound* (*mlb*) of Δ_1 and Δ_2 in $\langle \mathcal{DT}, \models^{\sharp/\flat} \rangle$ if for any lower bound Γ' of Δ_1 and Δ_2, $\Gamma' \models^{\sharp/\flat} \Gamma$ implies $\Gamma \models^{\sharp/\flat} \Gamma'$.

In the following, for any set X, let $min(X) = \{ x \in X \mid \neg\exists y \in X. y \subset x \}$ and $max(X) = \{ x \in X \mid \neg\exists y \in X. x \subset y \}$. We often denote $min\, X$ and $max\, X$ by omitting $()$. The proof of the next theorem is lengthy and is thus omitted because of the space limitation.

Theorem 2 *Let Δ_1, Δ_2 and Γ be default theories.*
(1) Γ *is an mub of* Δ_1 *and* Δ_2 *in* $\langle \mathcal{DT}, \models^{\sharp} \rangle$ *iff*

$$Ext(\Gamma) = min\{ cl(E \cup F) \mid E \in Ext(\Delta_1),\ F \in Ext(\Delta_2) \}.$$

(2) Γ *is an mlb of* Δ_1 *and* Δ_2 *in* $\langle \mathcal{DT}, \models^{\sharp} \rangle$ *iff*

$$Ext(\Gamma) = min(Ext(\Delta_1) \cup Ext(\Delta_2)).$$

(3) Γ *is an mub of* Δ_1 *and* Δ_2 *in* $\langle \mathcal{DT}, \models^{\flat} \rangle$ *iff*

$$Ext(\Gamma) = max(Ext(\Delta_1) \cup Ext(\Delta_2)).$$

(4) Γ *is an mlb of* Δ_1 *and* Δ_2 *in* $\langle \mathcal{DT}, \models^{\flat} \rangle$ *iff*

$$Ext(\Gamma) = max\{ E \cap F \mid E \in Ext(\Delta_1),\ F \in Ext(\Delta_2) \}.$$ [2]

Now, we can construct a poset from the pre-order set $\langle \mathcal{DT}, \models^{\sharp/\flat} \rangle$ in the usual way as follows. For any default theory $\Delta \in \mathcal{DT}$, consider the equivalence class:

$$[\Delta] = \{ \Gamma \in \mathcal{DT} \mid Ext(\Gamma) = Ext(\Delta) \},$$

and respectively define:

$$[\Delta] \succeq^{\sharp/\flat} [\Gamma] \quad \text{if} \quad \Delta \models^{\sharp/\flat} \Gamma.$$

We denote the equivalence classes from $\langle \mathcal{DT}, \models^{\sharp/\flat} \rangle$ as $\mathbf{DT}^{\sharp/\flat}$. Then, the relation $\succeq^{\sharp/\flat}$ is a partial order on $\mathbf{DT}^{\sharp/\flat}$.

Proposition 1 *The poset $\langle \mathbf{DT}^{\sharp/\flat}, \succeq^{\sharp/\flat} \rangle$ constitutes a complete lattice. The top element $\top^{\sharp}$ of $\langle \mathbf{DT}^{\sharp}, \succeq^{\sharp} \rangle$ is the class of incoherent default theories and the bottom element $\bot^{\sharp}$ of $\langle \mathbf{DT}^{\sharp}, \succeq^{\sharp} \rangle$ is $[\emptyset]$. The top element $\top^{\flat}$ of $\langle \mathbf{DT}^{\flat}, \succeq^{\flat} \rangle$ is the class of contradictory theories and the bottom element $\bot^{\flat}$ is the class of incoherent default theories.*

Connection with Entailment Relations

In existing studies on generality in first-order logic, the amount of information brought by a theory has been measured by the set of logical formulas entailed by the theory. That is, given two first-order theories T_1 and T_2, T_1 is defined as *more general* than T_2 if $T_1 \models T_2$ (Niblett 1988). The next theorem shows that $\models^{\sharp/\flat}$ in this paper extends the classical generality relation for first-order theories.

Theorem 3 *Let T_1 and T_2 be first-order theories. Then, $T_1 \models T_2$ iff $T_1 \models^{\sharp} T_2$ iff $T_1 \models^{\flat} T_2$.*

Proof. $T_1 \models T_2$ iff $cl(T_2) \subseteq cl(T_1)$. A first-order theory T is identified with the justification-free default theory $\Delta = \{ \frac{:}{\psi} \mid \psi \in T \}$ which has the single extension $cl(T)$. Hence, the result holds by the definition of $\models^{\sharp/\flat}$. □

[2] Note that $cl(E \cap F) = E \cap F$ holds in Theorem 2 (4).

In the presence of defaults with non-empty justifications and/or disjunctive defaults, however, a default theory generally produces multiple extensions. In this case, a connection to classical entailment is no longer applied. We then connect the generality relations over extensions with skeptical and credulous entailment in default reasoning. As a result, we will see that our orderings of $\sharp/\flat$-generality are also reasonable from the viewpoint of entailment relations.

Definition 4 Let Δ be a default theory and ψ a formula. Then, ψ is a *skeptical consequence* of Δ if ψ is included in every extension of Δ. ψ is a *credulous consequence* of Δ if ψ is included in some extension of Δ. The sets of skeptical and credulous consequences of Δ are denoted as $skp(\Delta)$ and $crd(\Delta)$, respectively.

Proposition 2 *If Δ is consistent, then*

$$skp(\Delta) = \bigcap_{E \in Ext(\Delta)} E, \qquad crd(\Delta) = \bigcup_{E \in Ext(\Delta)} E.$$

If Δ is incoherent, then $skp(\Delta) = \mathcal{L}$ and $crd(\Delta) = \emptyset$. If Δ is contradictory, then $skp(\Delta) = crd(\Delta) = \mathcal{L}$.

Theorem 4 *Let Δ_1 and Δ_2 be default theories.*

(1) *If $\Delta_1 \models^{\sharp} \Delta_2$ then $skp(\Delta_2) \subseteq skp(\Delta_1)$.*

(2) *$\Delta_1 \models^{\flat} \Delta_2$ iff $crd(\Delta_2) \subseteq crd(\Delta_1)$.*

Proof. (1) Assume that $\Delta_1 \models^{\sharp} \Delta_2$. If Δ_1 is inconsistent, then $skp(\Delta_1) = \mathcal{L}$ and thus $skp(\Delta_2) \subseteq skp(\Delta_1)$. Suppose that Δ_1 is consistent and $\psi \in skp(\Delta_2)$. Then, $\psi \in E$ for every extension $E \in Ext(\Delta_2)$. By $\Delta_1 \models^{\sharp} \Delta_2$, for any $E' \in Ext(\Delta_1)$, there is an extension $E'' \in Ext(\Delta_2)$ such that $E'' \subseteq E'$. Since $\psi \in E''$, $\psi \in E'$ too. That is, $\psi \in skp(\Delta_1)$. Hence, $skp(\Delta_2) \subseteq skp(\Delta_1)$.

(2) The only-if part can be proved in a similar way as (1). To see the if part, suppose $\Delta_1 \not\models^{\flat} \Delta_2$. Then, $Ext(\Delta_1) \neq \{\mathcal{L}\}$ and $Ext(\Delta_2) \neq \emptyset$. Then, there is an extension $F \in Ext(\Delta_2)$ such that $F \not\subseteq E$ for any $E \in Ext(\Delta_1)$. That is, for each $E \in Ext(\Delta_1)$, we can pick one formula $\varphi_E \in (F \setminus E)$. Then, $S = \{\varphi_E \mid E \in Ext(\Delta_1)\} \subseteq F$, whereas $S \not\subseteq E$ for any $E \in Ext(\Delta_1)$. Hence, $S \subseteq crd(\Delta_2)$ and $S \not\subseteq crd(\Delta_1)$, and thus $crd(\Delta_2) \not\subseteq crd(\Delta_1)$. □

By Theorem 4, (1) the more $\sharp$-general a theory is, the more it entails skeptically, and (2) the more $\flat$-general a theory is, the more it entails credulously. That is, the Smyth and Hoare orderings over default theories reflect the amount of information by skeptical and credulous entailment, respectively. Moreover, the $\flat$-generality precisely reflects informativeness of credulous entailment. On the other hand, the converse of Theorem 4 (1) does not hold for the $\sharp$-generality. [3]

Example 4 In Example 3, $Ext(\Delta_3) = \{cl(\{p\}), cl(\{q\})\}$ and $Ext(\Delta_4) = \{cl(\{p, q\})\}$. Then, $skp(\Delta_3) = cl(\{p \vee q\})$, $crd(\Delta_3) = cl(\{p\}) \cup cl(\{q\})$, and $skp(\Delta_4) = crd(\Delta_4) = cl(\{p, q\})$. Here, $\Delta_4 \models^{\sharp} \Delta_3$ and $\Delta_4 \models^{\flat} \Delta_3$,

[3] This asymmetry differs from the relations under the answer set semantics for logic programs, in which the converse of each property in (Inoue and Sakama 2006, Theorem 4.1) does not hold.

and correspondingly $skp(\Delta_3) \subset skp(\Delta_4)$ and $crd(\Delta_3) \subset crd(\Delta_4)$,[4] which verify Theorem 4.

To see that the converse of Theorem 4 (1) does not hold, let $\Delta_5 = \{p \vee q\}$ and $\Delta_6 = \{\, p \mid q \mid r \,\}$. Then, $skp(\Delta_6) = cl(\{p \vee q \vee r\}) \subset cl(\{p \vee q\}) = skp(\Delta_5)$, but $\Delta_5 \not\models^\sharp \Delta_6$.

Strong Generality in Default Logic

In this section, we investigate context-sensitive versions of $\flat/\sharp/\natural$-generality, which can be contrasted with the notion of strong equivalence (Turner 2001) in default logic.

Definition 5 Let Δ_1 and Δ_2 be default theories. Δ_1 is *strongly more $\sharp$-general than* Δ_2 (written $\Delta_1 \trianglerighteq^\sharp \Delta_2$) if $\Delta_1 \cup \Pi \models^\sharp \Delta_2 \cup \Pi$ for any default theory Π. Similarly, Δ_1 is *strongly more $\flat$-general than* Δ_2 (written $\Delta_1 \trianglerighteq^\flat \Delta_2$) if $\Delta_1 \cup \Pi \models^\flat \Delta_2 \cup \Pi$ for any default theory Π.

We write $\Delta_1 \trianglerighteq^\natural \Delta_2$ if $\Delta_1 \trianglerighteq^\sharp \Delta_2$ and $\Delta_1 \trianglerighteq^\flat \Delta_2$. We also write $\trianglerighteq^{\sharp/\flat}$ to represent either $\trianglerighteq^\sharp$ or $\trianglerighteq^\flat$. Then, $\langle \mathcal{DT}, \trianglerighteq^{\sharp/\flat} \rangle$ is a pre-ordered set. Obviously, if $\Delta_1 \trianglerighteq^{\sharp/\flat} \Delta_2$ then $\Delta_1 \models^{\sharp/\flat} \Delta_2$, respectively. Strong generality and strong equivalence have the following relationship.

Theorem 5 *Let Δ_1 and Δ_2 be default theories. Then, the following three are equivalent:*

(1) $\Delta_1 \trianglerighteq^\sharp \Delta_2$ *and* $\Delta_2 \trianglerighteq^\sharp \Delta_1$*;*

(2) $\Delta_1 \trianglerighteq^\flat \Delta_2$ *and* $\Delta_2 \trianglerighteq^\flat \Delta_1$*;*

(3) Δ_1 *and* Δ_2 *are strongly equivalent.*

Proof. $\Delta_1 \trianglerighteq^{\sharp/\flat} \Delta_2$ and $\Delta_2 \trianglerighteq^{\sharp/\flat} \Delta_1$ iff $\Delta_1 \cup \Pi \models^{\sharp/\flat} \Delta_2 \cup \Pi$ and $\Delta_2 \cup \Pi \models^{\sharp/\flat} \Delta_1 \cup \Pi$ for any default theory Π
iff $\Delta_1 \cup \Pi$ and $\Delta_2 \cup \Pi$ are weakly equivalent for any default theory Π (by Theorem 1)
iff Δ_1 and Δ_2 are strongly equivalent. □

Example 5 In Example 3, $\Delta_1 \trianglerighteq^\sharp \Delta_2 \trianglerighteq^\sharp \Delta_3$ holds. However, $\Delta_4 \not\trianglerighteq^\sharp \Delta_1$ because $\Delta_1 \cup \Pi$ is incoherent for $\Pi = \{\frac{p:}{q}\}$, while $\Delta_4 \cup \Pi = \Delta_4$ and hence $\Delta_4 \cup \Pi \not\models^\sharp \Delta_1 \cup \Pi$. On the other hand, $\Delta_3 \trianglerighteq^\flat \Delta_2 \trianglerighteq^\flat \Delta_1$ holds under $\trianglerighteq^\flat$, but there is no $\trianglerighteq^\natural$ relation between any pair from $\Delta_1, \ldots, \Delta_4$.

In Example 5, $\Delta_2 \models^\sharp \Delta_3$ and $\Delta_3 \models^\sharp \Delta_2$, while $\Delta_2 \trianglerighteq^\sharp \Delta_3$ and $\Delta_3 \not\trianglerighteq^\sharp \Delta_2$. Strong generality can thus be used to distinguish weakly equivalent theories, and any default theory in the equivalence class $[\Delta]$ induced by $\trianglerighteq^{\sharp/\flat}$ is strongly equivalent to Δ by Theorem 5. A necessary condition for strong generality is given as follows.

Theorem 6 *Let Δ_1 and Δ_2 be default theories.*

(1) *If* $\Delta_1 \trianglerighteq^\sharp \Delta_2$ *then* $Ext(\Delta_1) \subseteq Ext(\Delta_2)$.

(2) *If* $\Delta_1 \trianglerighteq^\flat \Delta_2$ *then* $Ext(\Delta_2) \subseteq Ext(\Delta_1)$.

Proof. (1) Assume that $Ext(\Delta_1) \not\subseteq Ext(\Delta_2)$. Then, $D = Ext(\Delta_1) \setminus Ext(\Delta_2) \neq \emptyset$. If there is an extension $E \in D$ ($\subseteq Ext(\Delta_1)$) such that $E \subset E'$ for some $E' \in Ext(\Delta_2)$, then $\Delta_1 \not\models^\sharp \Delta_2$ holds and thus $\Delta_1 \not\trianglerighteq^\sharp \Delta_2$. Otherwise, every extension in D is not a subset of any extension of Δ_2. Then for any $E \in D$ and any $E' \in Ext(\Delta_2)$, we can pick one formula $\psi_{(E,E')}$ such that $\psi_{(E,E')} \in (E \setminus E')$. Let $\Pi(E) = \{\, \frac{:\neg\psi_{(E,E')}}{false} \mid E' \in Ext(\Delta_2) \,\}$. Then, E is an extension of $\Delta_1 \cup \Pi(E)$, but $\Delta_2 \cup \Pi(E)$ has no extension. Therefore, $\Delta_1 \cup \Pi(E) \not\models^\sharp \Delta_2 \cup \Pi(E)$, and hence $\Delta_1 \not\trianglerighteq^\sharp \Delta_2$ holds.

The claim (2) can be proved in a similar way. □

[4] Note that $crd(\Delta_3) \neq crd(\Delta_4)$ because $p \wedge q$ belongs to the latter set only. Again this differs from the relation under the answer set semantics for corresponding logic programs, in which only literal consequences are taken into account.

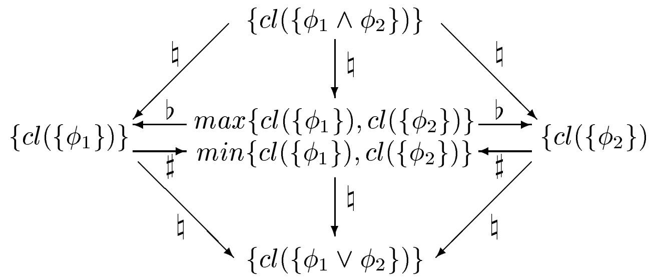

Figure 1: lg/gs of clauses ϕ_1 and ϕ_2

The converse of Theorem 6 does not hold because in Example 5, $Ext(\Delta_3) = Ext(\Delta_2)$ but $\Delta_3 \not\trianglerighteq^\sharp \Delta_2$.

Theorem 6 states that strong $\sharp/\flat$-generality implies *inclusion*. This is a rather unexpected result. It also follows that $\Delta_1 \trianglerighteq^\natural \Delta_2$ implies $Ext(\Delta_1) = Ext(\Delta_2)$.

Discussion

Theories of generality relations over first-order clauses have been studied in the field of *inductive logic programming* (Niblett 1988; Nienhuys-Cheng and de Wolf 1997; Plotkin 1970). Plotkin (1970) studies the generality relation between a pair of clauses and a generality order is introduced over a set of clauses. By contrast, we considered generality relations between *theories*. Then, generality relations between clauses can be captured as a special case of our $\sharp/\flat/\natural$-generality: *A clause ϕ is more general than a clause ϕ' iff* $\{\phi\} \models^\natural \{\phi'\}$ (see Theorem 3). Plotkin also defined the *least (general) generalization* (lg) of two clauses ϕ_1 and ϕ_2 (Plotkin 1970), which corresponds to an mub of $\{\phi_1\}$ and $\{\phi_2\}$ in $\langle \mathcal{DT}, \models^\sharp \rangle$ as long as $\phi_1 \wedge \phi_2$ is a clause. Similarly, the *greatest specialization* (gs) of clauses ϕ_1 and ϕ_2 can be given as $\phi_1 \vee \phi_2$, which is always a clause (Nienhuys-Cheng and de Wolf 1997) and exactly corresponds to an mlb of $\{\phi_1\}$ and $\{\phi_2\}$ in $\langle \mathcal{DT}, \models^\flat \rangle$. For example, when $\phi_1 = (p \rightarrow r)$ and $\phi_2 = (q \rightarrow r)$, Plotkin's lg of them is defined as r, but the least upper bound is $\phi_1 \wedge \phi_2 = ((p \vee q) \rightarrow r)$, which is not a clause. Both the gs and the greatest lower bound of them is $\phi_1 \vee \phi_2 = ((p \wedge q) \rightarrow r)$.

Figure 1 illustrates the relationships between lg/gs of ϕ_1 and ϕ_2 and mub/mlb of $\{\phi_1\}$ and $\{\phi_2\}$, which are induced by Theorem 2. In the figure, $E \longrightarrow F$ means that E is more general than F under the ordering in the label, and the label $\natural$ represents that both $\sharp$- and $\flat$-generality hold. Note in Figure 1 that under the ordering of $\sharp$-generality (resp. $\flat$-generality), an mlb (resp. mub) may consist of two extensions. This is a unique feature of generality relations in default logic. Under the $\natural$-generality ordering, however, $\{cl(\{\phi_1 \wedge \phi_2\})\}$ (resp. $\{cl(\{\phi_1 \vee \phi_2\})\}$) is the unique extension of any mub (resp. mlb). Besides these correspondences, the main contribution of this paper is a theory of generality relations for *nonmonotonic theories*, which has rarely been discussed in the field of inductive logic programming.

Frameworks to compare logic programs have recently been discussed in (Eiter *et al.* 2005; Inoue and Sakama 2006). Eiter *et al.* (2005) compare logic programs with respect to binary relations such as equivalence and inclusion on the double powerset of atoms, but do not consider generality relations. Inoue and Sakama (2006) define generality relations for logic programs, which can be captured as special cases of the theories presented in this paper as follows. An *extended disjunctive program* (EDP) (Gelfond and Lifschitz 1991) is a set of *rules* of the form:

$$L_1 \,;\, \cdots \,;\, L_l \;\leftarrow\; L_{l+1}, \,\ldots, \, L_m, \, not\, L_{m+1}, \,\ldots, \, not\, L_n$$

where $n \geq m \geq l \geq 0$ and each L_i is a literal, not is *negation as failure*, and ";" represents disjunction. Any EDP P can be embedded to a disjunctive default theory $\delta(P)$ obtained from P by replacing each rule r with the default:

$$\frac{L_{l+1} \wedge \cdots \wedge L_m : \neg L_{m+1}, \ldots, \neg L_n}{L_1 \mid \cdots \mid L_l}\,.$$

Then, S is an *answer set* of P iff $S = E \cap Lit$ for some $E \in Ext(\delta(P))$, where Lit is the set of all ground literals in the language of P (Gelfond *et al.* 1991). With this embedding, we have the correspondence: *An EDP P_1 is more $\sharp/\flat$-general than an EDP P_2 in the sense of (Inoue and Sakama 2006) iff $\delta(P_1) \models^{\sharp/\flat} \delta(P_2)$.* Compared with (Inoue and Sakama 2006), the formalization in this paper more naturally extends the generality relation in first-order logic. For example, the clause $\phi_1 = (p \rightarrow r)$ is more general than the clause $\phi_2 = (p \wedge q \rightarrow r)$ in Horn theories (Niblett 1988), and correspondingly we have $\{\phi_1\} \models^{\sharp/\flat} \{\phi_2\}$ by Theorem 3. On the other hand, the default embeddings of the rules $\varphi_1 = (r \leftarrow p)$ and $\varphi_2 = (r \leftarrow p, q)$ are weakly equivalent because $\{\varphi_1\}$ and $\{\varphi_2\}$ have the same answer sets $\{\emptyset\}$. Then, $\delta(\{\varphi_2\}) \models^{\sharp/\flat} \delta(\{\varphi_1\})$, while $\{\phi_2\} \not\models^{\sharp/\flat} \{\phi_1\}$.

Sakama (2005) defines an ordering over Reiter's default theories based on ten-valued logic. According to (Sakama 2005, Theorem 2.9), a default theory $\Delta_1 = (D_1, W_1)$ is *stronger* than $\Delta_2 = (D_2, W_2)$,[5] denoted as $\Delta_1 \geq_{DL} \Delta_2$, if (1) $W_1 \models W_2$, (2) $\forall E_1 \in Ext(\Delta_1)\, \exists E_2 \in Ext(\Delta_2).\, E_2 \subseteq E_1$, and (3) $\forall E_2 \in Ext(\Delta_2)\, \exists E_1 \in Ext(\Delta_1).\, E_2 \subseteq E_1$. Therefore, we have the following relation: *$\Delta_1 \geq_{DL} \Delta_2$ if $W_1 \models W_2$ and $\Delta_1 \models^{\natural} \Delta_2$.* Sakama distinguishes definite and credulous/skeptical default information derived from a theory. For example, $\{p\}$ is stronger than $\{\frac{:p}{p}\}$, in the sense that the definite consequence p in the former has a higher degree of truth than the default consequence p in the latter. We do not distinguish them in this paper; they are weakly equivalent. As a result, $\Delta_1 \geq_{DL} \Delta_2$ and $\Delta_2 \geq_{DL} \Delta_1$ imply $Ext(\Delta_1) = Ext(\Delta_2)$, but not vice versa. In contrast, every default theory in the equivalence class $[\Delta]$ induced by $\models^{\sharp/\flat}$ is weakly equivalent to Δ by Theorem 1.

[5] Sakama defines a default theory as a pair of non-disjunctive defaults (D) and first-order formulas (W) as in (Reiter 1980), and excludes justification-free defaults $\frac{\alpha:}{\gamma}$ from D.

We have developed the theory of generality relations from the semantical viewpoint in this paper. An important future topic is to develop computational aspects of generality. For another issue, strong generality in the current form is very strong, and it is meaningful to relax its context-dependent generality condition to a relativized one.

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